CHAPTER 1

Number Systems

Animation 1.1: Complex Plane
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1.1 Introduction

In the very beginning, human life was simple. An early ancient herdsman compared sheep (or cattle) of his herd with a pile of stones when the herd left for grazing and again on its return for missing animals. In the earliest systems probably the vertical strokes or bars such as I, II, III, llll etc.. were used for the numbers 1, 2, 3, 4 etc. The symbol "lllll" was used by many people including the ancient Egyptians for the number of fingers of one hand.

Around 5000 B.C, the Egyptians had a number system based on 10. The symbol 𓎆 for 10 and 𓍢 for 100 were used by them. A symbol was repeated as many times as it was needed. For example, the numbers 13 and 324 were symbolized as 𓎆||| and 𓍢𓍢𓍢𓎆𓎆|||| respectively. The symbol 𓍢𓍢𓍢𓎆𓎆|||| was interpreted as 100 + 100 +100+10+10+1+1+1 +1. Different people invented their own symbols for numbers. But these systems of notations proved to be inadequate with advancement of societies and were discarded. Ultimately the set {1, 2, 3, 4, ...} with base 10 was adopted as the counting set (also called the set of natural numbers). The solution of the equation $x + 2 = 2$ was not possible in the set of natural numbers, So the natural number system was extended to the set of whole numbers. No number in the set of whole numbers W could satisfy the equation $x + 4 = 2$ or $x + a = b$, if $a > b$, and $a, b, \in W$. The negative integers –1, –2, –3, ... were introduced to form the set of integers Z = {0, ±1, ±2 ,...).

Again the equation of the type $2x = 3$ or $bx = a$ where $a,b,\in Z$ and $b \neq 0$ had no solution in the set Z, so the numbers of the form $\frac{a}{b}$ where $a,b,\in Z$ and $b \neq 0$, were invented to remove such difficulties. The set $Q = \{\frac{a}{b} \mid a,b,\in Z \wedge b \neq 0\}$ was named as the set of rational numbers. Still the solution of equations such as $x^2 = 2$ or $x^2 = a$ (where a is not a perfect square) was not possible in the set Q. So the irrational numbers of the type $\pm\sqrt{2}$ or $\pm\sqrt{a}$ where a is not a perfect square were introduced. This process of enlargement of the number system ultimately led to the set of real numbers $\Re = Q \cup Q'$ (Q' is the set of irrational numbers) which is used most frequently in everyday life.

1.2 Rational Numbers and Irrational Numbers

We know that a rational number is a number which can be put in the form $\frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$. The numbers $\sqrt{16}$, 3.7, 4 etc., are rational numbers. $\sqrt{16}$ can be reduced to the form $\frac{p}{q}$ where $p, q \in Z$, and $q \neq 0$ because $\sqrt{16} = 4 = \frac{4}{1}$.

Irrational numbers are those numbers which cannot be put into the form $\frac{p}{q}$ where $p, q \in Z$ and $q \neq 0$. The numbers $\sqrt{2}$, $\sqrt{3}$, $\frac{7}{\sqrt{5}}$, $\sqrt{\frac{5}{16}}$ are irrational numbers.

1.2.1 Decimal Representation of Rational and Irrational Numbers

1) Terminating decimals: A decimal which has only a finite number of digits in its decimal part, is called a terminating decimal. Thus 202.04, 0.0000415, 100000.41237895 are examples of terminating decimals.

Since a terminating decimal can be converted into a common fraction, so every terminating decimal represents a **rational number**.

2) Recurring Decimals: This is another type of **rational numbers**. In general, a recurring or periodic decimal is a decimal in which one or more digits repeat indefinitely.

It will be shown (in the chapter on sequences and series) that a recurring decimal can be converted into a common fraction. So **every recurring decimal represents a rational number:**

A non-terminating, non-recurring decimal is a decimal which neither terminates nor it is recurring. It is not possible to convert such a decimal into a common fraction. Thus a **non-terminating, non-recurring decimal represents an irrational number.**

Example 1:

i) .25 $(=\frac{25}{100})$ is a rational number.

ii) .333...$(=\frac{1}{3})$ is a recurring decimal, it is a rational number.

iii) $2.\bar{3}(=2.333...)$ is a rational number.

iv) 0.142857142857... $(=\frac{1}{7})$ is a rational number.

v) 0.01001000100001 ... is a non-terminating, non-periodic decimal, so it is an irrational number.

vi) 214.121122111222 1111 2222 ... is also an irrational number.

vii) 1.4142135 ... is an irrational number.

viii) 7.3205080 ... is an irrational number.

ix) 1.709975947 ... is an irrational number.

x) 3.141592654... is an important irrational number called it π(Pi) which denotes the constant ratio of the circumference of any circle to the length of its diameter i.e.,

$$\pi = \frac{\text{circumference of any circle}}{\text{length of its diameter.}}$$

An approximate value of π is $\frac{22}{7}$,a better approximation is $\frac{355}{113}$ and a still better approximation is 3.14159. The value of π correct to 5 lac decimal places has been determined with the help of computer.

Example 2: Prove $\sqrt{2}$ is an irrational number.

Solution: Suppose, if possible, $\sqrt{2}$ is rational so that it can be written in the form *p/q* where *p,q* $\in$ Z and $q \neq 0$. Suppose further that p/q is in its lowest form.

Then $\sqrt{2}$ = *p/q*, $(q \neq 0)$

Squaring both sides we get;

$$2 = \frac{p^2}{q^2} \text{ or } p^2 = 2q^2 \qquad (1)$$

The R.H.S. of this equation has a factor 2. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in the square. Therefore, p^2 should be of the form $4p'^2$ so that equation (1) takes the form:

$$4p'^2 = 2q^2 \qquad ....(2)$$

i.e., $$2p'^2 = q^2 \qquad ....(3)$$

In the last equation, 2 is a factor of the L.H.S. Therefore, q^2 should be of the form $4q'^2$ so that equation 3 takes the form

$$2p'^2 = 4q'^2 \quad \text{i.e., } p'^2 = 2q'^2 \qquad ....(4)$$

From equations (1) and (2),

$$p = 2p'$$

and from equations (3) and (4)

$$q = 2q'$$

$$\therefore \quad \frac{p}{q} = \frac{2p'}{2q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form. Hence $\sqrt{2}$ is irrational.

Example 3: Prove $\sqrt{3}$ is an irrational number.

Solution: Suppose, if possible $\sqrt{3}$ is rational so that it can be written in the form *p/q* when *p,q* $\in$ *Z* and $q \neq 0$. Suppose further that *p/q* is in its lowest form,

then $\sqrt{3}$ = *p/q* , $(q \neq 0)$

Squaring this equation we get;

$$3 = \frac{p^2}{q^2} \qquad \text{or } p^2 = 3q^2 \qquad ....(1)$$

The R.H.S. of this equation has a factor 3. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in the square. Therefore, p^2 should be of the form $9p'^2$ so that equation (1) takes the form:

$9p'^2 = 3q^2$ (2)

i.e., $3p'^2 = q^2$ (3)

In the last equation, 3 is a factor of the L.H.S. Therefore, q^2 should be of the form $9q'^2$ so that equation (3) takes the form

$3p'^2 = 9q^2$ i.e., $p'^2 = 3q'^2$ (4)

From equations (1) and (2),

$P = 3P'$

and from equations (3) and (4)

$q = 3q'$

$$\therefore \quad \frac{p}{q} = \frac{3p'}{3q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form.

Hence $\sqrt{3}$ is irrational.

Note: Using the same method we can prove the irrationality of $\sqrt{5}, \sqrt{7}, \ldots, \sqrt{n}$ where n is any prime number.

1.3 Properties of Real Numbers

We are already familiar with the set of real numbers and most of their properties. We now state them in a unified and systematic manner. Before stating them we give a preliminary definition.

Binary Operation: A binary operation may be defined as a function from A $\times$ A into A, but for the present discussion, the following definition would serve the purpose. A *binary operation* in a set A is a rule usually denoted by * that assigns to any pair of elements of A, taken in a definite order, another element of A.

Two important binary operations are addition and multiplication in the set of real numbers. Similarly, union and intersection are binary operations on sets which are subsets of the

same Universal set.

$\Re$ usually denotes the set of real numbers. We assume that two binary operations addition (+) and multiplication (. or x) are defined in $\Re$. Following are the properties or laws for real numbers.

1. Addition Laws: -

i) Closure Law of Addition

$\forall$ a, b$\in\Re$, a + b$\in\Re$ ($\forall$ stands for "for all")

ii) Associative Law of Addition

$\forall$ a, b,c$\in\Re$, a + (b + c) = (a + b) + c

iii) Additive Identity

$\forall$ $a\in\Re$, $\exists$ $0\in\Re$ such that $a + 0 = 0 + a = a$

($\exists$ stands for "there exists").

0(read as zero) is called the identity element of addition.

iv) Additive Inverse

$\forall$ $a\in\Re$, $\exists$ $(-a)\in\Re$ such that

$a + (-a) = 0 = (-a) + a$

v) Commutative Law for Addition

$\forall$ a, b$\in\Re$, a + b = b + a

2. Multiplication Laws

vi) Closure I.aw of Multiplication

$\forall$ a, b$\in\Re$, a. b$\in\Re$ (a,b is usually written as ab).

vii) Associative Law for Multiplication

$\forall$ $a, b, c\in\Re$, $a(bc) = (ab)c$

viii) Multiplicative Identity

$\forall$ $a\in\Re$, $\exists$ $1\in\Re$ such that $a.1 = 1.a = a$

1 is called the multiplicative identity of real numbers.

ix) Multiplicative Inverse

$\forall$ $a(\neq 0)\in\Re$, $\exists$ $a^{-1}\in\Re$ such that $a.a^{-1} = a^{-1}.a = 1$ (a^{-1} is also written as $\frac{1}{a}$).

x) Commutative Law of multiplication

$\forall$ $a, b\in\Re$, $ab = ba$

3. Multiplication – Addition Law

xi) $\forall$ ***a, b, c*** $\in \Re$,

$a(b + c) = ab + ac$ (Distrihutivity of multiplication over addition).

$(a + b)c = ac + bc$

In addition to the above properties $\Re$ possesses the following properties.

i) Order Properties (described below).

ii) Completeness axiom which will be explained in higher classes.

The above properties characterizes $\Re$ i.e., only $\Re$ possesses all these properties.

Before stating the order axioms we state the properties of equality of numbers.

4. Properties of Equality

Equality of numbers denoted by "=" possesses the following properties:-

i) Reflexive property $\forall\ a \in \Re,\ a = a$

ii) Symmetric Property $\forall\ a,b \in \Re,\ a = b \Rightarrow b = a$.

iii) Transitive Property $\forall\ a,b,c \in \Re,\ a = b \wedge b = c \Rightarrow a = c$

iv) Additive Property $\forall\ a,b,c \in \Re,\ a = b \Rightarrow a + c = b + c$

v) Multiplicative Property $\forall\ a,b,c \in \Re,\ a = b \Rightarrow ac = bc \wedge ca = cb$.

vi) Cancellation Property w.r.t. addition

$\forall\ a,b,c \in \Re,\ a + c = b + c \Rightarrow a = b$

vii) Cancellation Property w.r.t. Multiplication:

$\forall\ a,b,c \in \Re,\ ac = bc \Rightarrow a = b,\ c \neq 0$

5. Properties of Ineualities (Order properties)

1) Trichotomy Property $\forall\ a,b \in \Re$

either $a = b$ or $a > b$ or $a < b$

2) Transitive Property $\forall\ a,b,c \in \Re$

i) $a > b \wedge b > c \Rightarrow a > c$ ii) $a < b \wedge b < c \Rightarrow a < c$

3) Additive Property: $\forall\ a,b,c \in \Re$

a) i) $a > b \Rightarrow a + c > b + c$ b) i) $a > b \wedge c > d \Rightarrow a+c > b+d$

ii) $a < b \Rightarrow a + c < b + c$ ii) $a < b \wedge c < d \Rightarrow a+c < b+d$

4) Multiplicative Properties:

a) $\forall\ a,b,c \in \Re$ and $c > 0$

i) $a > b \Rightarrow ac > bc$ ii) $a < b \Rightarrow ac < bc$.

b) $\forall\ a,b,c \in \Re$ and $c < 0$.

i) $a > b \Rightarrow ac < bc$ ii) $a < b \Rightarrow ac > bc$

c) $\forall\ a,b,c,d \in \Re$ and a,b,c,d are all positive.

i) $a > b \wedge c > d \Rightarrow ac > bd$. ii) $a < b \wedge c < d \Rightarrow ac < bd$

Note That:

1. Any set possessing all the above 11 properties is called a field.
2. From the multiplicative properties of inequality we conclude that: - If both the sides of an inequality are multiplied by a +ve number, its direction does not change, but multiplication of the two sides by -ve number reverses the direction of the inequality.
3. a and $(-a)$ are additive inverses of each other. Since by definition inverse of $-a$ is a,

$$\therefore\ -(-a) = a$$

4. The left hand member of the above equation should be read as negative of 'negative a' and not 'minus minus a'.
5. a and $\frac{1}{a}$ are the multiplicative inverses of each other. Since by definition inverse of $\frac{1}{a}$ is a (i.e., inverse of a^{-1} is a), $a \neq 0$

$$\therefore\ (a^{-1})^{-1} = a \quad \text{or} \quad \frac{1}{\frac{1}{a}} = a$$

Example 4: Prove that for any real numbers a, b

i) $a.0 = 0$ ii) $ab = 0 \Rightarrow a = 0 \vee b = 0$ [$\vee$ stands for "or"]

Solution: i) $a.0 = a[1 + (-1)]$ (Property of additive inverse)

$= a(1 - 1)$ (Def. of subtraction)

$= a.1 - a.1$ (Distributive Law)

$= a - a$ (Property of multiplicative identity)

$= a + (-a)$ (Def. of subtraction)

$= 0$ (Property of additive inverse)

Thus $a.0 = 0$.

ii) Given that $ab = 0$ (1)

Suppose $a \neq 0$, then exists

(1) gives: $\frac{1}{a}(ab) = \frac{1}{a}.0$ (Multiplicative property of equality)

$\Rightarrow (\frac{1}{a}.a)b = \frac{1}{a}.0$ (Assoc. law of $\times$)

$\Rightarrow 1.b = 0$ (Property of multiplicative inverse).

$\Rightarrow b = 0$ (Property of multiplicative identity).

Thus if $ab = 0$ and $a \neq 0$, then $b = 0$

Similarly it may be shown that

if $ab = 0$ and $b \neq 0$, then $a = 0$.

Hence $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Example 5: For real numbers a,b show the following by stating the properties used.

i) $(-a)\,b = a\,(-b) = -ab$ ii) $(-a)\,(-b) = ab$

Solution: i) $(-a)(b) + ab = (-a + a)b$ (Distributive law)

$= 0.b = 0.$ (Property of additive inverse)

$\therefore\ (-a)b + ab = 0$

i.e.. $(-a)b$ and ab are additive inverse of each other.

$\therefore (-a)b = -(ab) = -ab$ (Θ $-(ab)$ is written as $-ab$)

ii) $(-a)\,(-b) - ab = (-a)(-b) + (-ab)$

$= (-a)(-b) + (-a)(b)$ (By (i))

$= (-a)(-b + b)$ (Distributive law)

$= (-a)\,.0 = 0.$ (Property of additive inverse)

$(-a)(-b) = ab$

Example 6: Prove that

i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$ (Principle for equality of fractions

ii) $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

iii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ (Rule for product of fractions).

iv) $\frac{a}{b} = \frac{ka}{kb}, (k \neq 0)$ (Golden rule of fractions)

v) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ (Rule for quotient of fractions).

The symbol $\Leftrightarrow$ stands for iff i.e.. if and only if.

Solution:

i) $\frac{a}{b} \Rightarrow \frac{c}{d} \quad \frac{a}{b}(\not{=}bd) \quad \frac{c}{d}(bd)$

$\Rightarrow \frac{a.1}{b}(bd) = \frac{c.1}{d}(bd)$

$\Rightarrow a.(\frac{1}{b}.b).d = c.(\frac{1}{d}.bd)$

$= c(bd.\frac{1}{d})$

$\Rightarrow ad = cb$

$\therefore\ ad = bc$

Again $ad = bc \Rightarrow (ad) \times \frac{1}{b}.\frac{1}{d} = b.c.\frac{1}{b}.\frac{1}{d}$

$\Rightarrow a.\frac{1}{b}.d\frac{1}{d} = b.\frac{1}{b}\ .\ c.\frac{1}{d}$

$\Rightarrow \frac{a}{b} = \frac{c}{d}.$

ii) $(ab).\frac{1}{a}.\frac{1}{b} = (a.\frac{1}{a}).(b\frac{1}{b}) = 1.1 = 1$

Thus ab and $\frac{1}{a}.\frac{1}{b}$ are the multiplicative inverse of each other. But multiplicative inverse of ab is $\frac{1}{ab}$

$\therefore \frac{1}{ab} = \frac{1}{a}.\frac{1}{b}.$

iii) $\frac{a}{b}.\frac{c}{d} = (a.\frac{1}{b}).(c.\frac{1}{d})$

$=(ac)(\frac{1}{b}.\frac{1}{d})$ (Using commutative and associative laws of multiplication)

$=ac.\frac{1}{bd}=\frac{ac}{bd}.$

$=\frac{a}{b}.\frac{c}{d}=\left|\frac{ac}{bd}\right|$

iv) $\frac{a}{b}=\frac{a}{b}.1=\frac{a}{b}.\frac{k}{k}=\frac{ak}{ak}$

$\therefore \frac{a}{b}=\frac{ak}{bk}.$

v) $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b}(bd)}{\frac{c}{d}(bd)}=\frac{ad(\frac{1}{b}.b)}{cb(\frac{1}{d}.d)}=\frac{ad}{bc}.$

Example 7: Does the set {1, –1 } possess closure property with respect to
i) addition ii) multiplication?

Solution: i) 1 + 1 = 2, 1 + (–1) = 0 = –1 + 1
–1 + (–1) = –2

But 2, 0, –2 do not belong to the given set. That is, all the sums do not belong to the given set. So it does not possess closure property w.r.t. addition.

ii) 1.1= 1, 1. (–1) = –1, (–1) .1 = –1, (–1). (–1) = 1
Since all the products belong to the given set, it is closed w.r.t multiplication.

Exercise 1.1

1. Which of the following sets have closure property w.r.t. addition and multiplication?

i) {0} ii) {1} iii) (0 , – 1) iv) { 1, –1 }

2. Name the properties used in the following equations.
(Letters, where used, represent real numbers).

i) 4 + 9 = 9 + 4
ii) $(a+1)+\frac{3}{4}=a+(1+\frac{3}{4})$
iii) $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$
iv) 100 + 0 = 100
v) 1000 × 1 = 1000
vi) 4.1 + (– 4 .1) = 0
vii) $a - a = 0$
viii)
ix) $a(b - c) = ab - ac$
x) $(x - y)z = xz - yz$
xi) 4×(5 × 8) = (4 × 5) × 8
xii) $a(b + c - d) = ab + ac - ad$.

3. Name the properties used in the following inequalities:

i) $-3 < -2 \Rightarrow 0 < 1$
ii) $-5 < -4 \Rightarrow 20 > 16$
iii) $1 > -1 \Rightarrow -3 > -5$
iv) $a < 0 \Rightarrow -a > 0$
v) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$
vi) $a > b \Rightarrow -a < -b$

4. Prove the following rules of addition: -

i) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$
ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

5. Prove that $-\frac{7}{12}-\frac{5}{18}=\frac{-21-10}{36}$

6. Simplify by justifying each step: -

i) $\frac{4+16x}{4}$
ii) $\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

iii) $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$ iv) $\dfrac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}$

1.4 Complex Numbers

The history of mathematics shows that man has been developing and enlarging his concept of **number** according to the saying that "Necessity is the mother of invention". In the remote past they stared with the set of counting numbers and invented, by stages, the negative numbers, rational numbers, irrational numbers. Since square of a positive as well as negative number is a positive number, the square root of a negative number does not exist in the realm of real numbers. Therefore, square roots of negative numbers were given no attention for centuries together. However, recently, properties of numbers involving square roots of negative numbers have also been discussed in detail and such numbers have been found useful and have been applied in many branches o f pure and applied mathematics. The numbers of the form $x + iy$, where $x, y \in \Re$, and i = ,are called **complex numbers,** here x is called **real part** and y is called **imaginary part** of the complex number. For example, 3 + 4i, 2 – i etc. are complex numbers.

Note: Every real number is a complex number with 0 as its imaginary part.

Let us start with considering the equation.

$$x^2 + 1 = 0 \qquad (1)$$

$$\Rightarrow \quad x^2 = -1$$

$$\Rightarrow \quad x = \pm\sqrt{-1}$$

$\sqrt{-1}$ does not belong to the set of real numbers. We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota).

The product of a real number and i is also an **imaginary number**

Thus $2i$, $-3i$, $\sqrt{5}i$, $-\frac{11}{2}i$ are all imaginary numbers, i which may be written $1.i$ is also an imaginary number.

Powers of i :

$i^2 = -1$ (by defination)

$i^3 = i^2.i = -1.i = -i$

$i^4 = i^2 \times i^2 = (-1)(-1) = 1$

Thus any power of i must be equal to 1, i,–1 or $-i$. For instance,

$i^{13} = (i^2)^6.^2 = (-1)^6.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$ etc.

1.4.1 Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions. The symbols a,b,c,d,k, where used, represent real numbers.

1) $a + bi = c + di \Rightarrow a = c \;\; b = d$.

2) Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

3) $k(a + bi) = ka + kbi$

4) $(a + bi) - (c + di) = (a + bi) + [-(c + di)]$
$= a + bi + (-c - di)$
$= (a - c) + (b - d)i$

5) $(a + bi).(c + di) = ac + adi + bci + bdi = (ac - bd) + (ad + bc)i$.

6) Conjugate Complex Numbers: Complex numbers of the form (a + bi) and (a – bi) which have the same real parts and whose imaginary parts differ in sign only, are called conjugates of each other. Thus 5 + 4i and 5 – 4i, –2 + 3i and – 2 – 3i,– $\sqrt{5}\,i$ and $\sqrt{5}\,i$ are three pairs of conjugate numbers.

Note: A real number is self-conjugate.

1.4.2 Complex Numbers as Ordered Pairs of Real Numbers

We can define complex numbers also by using ordered pairs. Let C be the set of ordered pairs belonging to $\Re \times \Re$ which are subject to the following properties: -

i) $(a,b)=(c,d) \Leftrightarrow a=c \wedge b=d.$
ii) $(a, b) + (c, d) = (a + c, b + d)$
iii) If k is any real number, then $k(a, b) = (ka, kb)$
iv) $(a, b)(c, d) = (ac - bd, ad + bc)$

Then C is called the set of *complex numbers*. It is easy to sec that $(a, b) - (c, d) = (a - c, b - d)$

Properties (1), (2) and (4) respectively define equality, sum and product of two complex numbers. Property (3) defines the product of a real number and a complex number.

Example 1: Find the sum, difference and product of the complex numbers (8, 9) and (5, –6)

Solution: Sum = (8 + 5, 9 – 6) = (13, 3)
Difference = (8 – 5, 9 – (–6)) = (3, 15)
Product = (8.5 – (9)(–6), 9.5 + (–6) 8)
= (40 + 54, 45 – 48)
= (94, –3)

1.4.3 Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties 1(i to v), 2(vi to x) and 3(xi) of Art. 1.3.

By way of explanation of some points we observe as follows:-

i) The additive identity in C is (0, 0).
ii) Every complex number (a, b) has the additive inverse $(-a, -b)$ i.e., $(a, b) + (-a, -b) = (0, 0)$.
iii) The multiplicative identity is (1, 0) i.e.,
$(a, b).(1, 0) = (a.1 - b.0, b.1 + a.0) = (a, b)$.
$= (1, 0)\,(a, b)$
iv) Every non-zero complex number {i.e., number not equal to (0, 0)} has a multiplicative inverse.

The multiplicative inverse of (a, b) is $\left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

$$(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0), \text{ the identity element}$$
$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a,b)$$

v) $(a, b)[(c, d) \pm (e, f)] = (a, b)(c, d) \pm (a, b)(e, f)$

Note: The set C of complex numbers does not satisfy the order axioms. In fact there is no sense in saying that one complex number is greater or less than another.

1.4.4 A Special Subset of *C*

We consider a subset of C whose elements are of the form $(a, 0)$ i.e., second component of each element is zero.

Let $(a, 0)$, $(c, 0)$ be two elements of this subset. Then

i) $(a, 0) + (c, 0) = (a + c, 0)$ ii) $k(a, 0) = (ka, 0)$
iii) $(a, 0) \times (c, 0) = (ac, 0)$
iv) Multiplicative inverse of $(a, 0)$ is $\left(\dfrac{1}{a}, 0\right), a \neq 0.$

Notice that the results are the same as we should have obtained if we had operated on the real numbers a and c ignoring the second component of each ordered pair i.e., 0 which has played no part in the above calculations.

On account of this special feature wc identify the complex number $(a, 0)$ with the real number a i.e., we postulate:

$(a, 0)= a$ (1)
Now consider (0, 1)
$(0, 1) . (0, 1) = (-1, 0)$
$= -1$ (by (1) above).

If we set $(0, 1) = i$ (2)
then $(0, 1)^2 = (0,1)(0,1) = i.i = i^2 = -1$

We are now in a position to write every complex number given as an ordered pair, in terms of i. For example

$(a, b) = (a, 0) + (0, b)$ (def. of addition)

$= a(1, 0) + b(0, 1)$ (by (1) and (2) above)

$= a.1 + bi$

$= a + ib$

Thus $(a, b) = a + ib$ where $i^2 = -1$

This result enables us to convert any Complex number given in one notation into the other.

Exercise 1.2

1. Verify the addition properties of complex numbers.

2. Verify the multiplication properties of the complex numbers.

3. Verify the distributive law of complex numbers.

$(a, b)[(c, d) + (e, f)] = (a, b)(c, d) + (a, b)(e, f)$

(**Hint:** Simplify each side separately)

4. Simplify' the following:

i) i^9 ii) i^{14} iii) $(-i)^{19}$ iv) $(-\frac{21}{2})$

5. Write in terms of i

i) $\sqrt{-1b}$ ii) $\sqrt{-5}$ iii) $\sqrt{\frac{-16}{25}}$ iv) $\sqrt{\frac{1}{-4}}$

Simplify the following:

6. (7, 9) + (3, –5) **7.** (8, –5) – (–7, 4) **8.** (2, 6)(3, 7)

9. (5, –4) (–3, –2) **10.** (0, 3) (0, 5) **11.** (2, 6)÷(3, 7).

12. (5, –4) ÷(–3, –8) $\left(\text{Hint for 11: } \frac{(2,6)}{(3,7)} = \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i} \text{ etc.}\right)$

13. Prove that the sum as well as the product of any two conjugate complex numbers is a real number.

14. Find the multiplicative inverse of each of the following numbers:

i) (–4, 7) ii) $\left(\sqrt{2}, -\sqrt{5}\right)$ iii) (1, 0)

15. Factorize the following:

i) $a^2 + 4b^2$ ii) $9a^2 + 16b^2$ iii) $3x^2 + 3y^2$

16. Separate into real and imaginary parts (write as a simple complex number): -

i) $\frac{2-7i}{4+5i}$ ii) $\frac{(-2+3i)^2}{(1+i)}$ iii) $\frac{i}{1+i}$

1.5 The Real Line

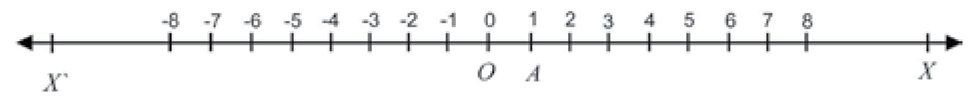

Fig. (1)

In Fig.(1), let $\overleftrightarrow{X'X}$ be a line. We represent the number 0 by a point O (called the origin) of the line. Let $|OA|$ represents a unit length. According to this unit, positive numbers are represented on this line by points to the right of O and negative numbers by points to the left of O. It is easy to visualize that all +ve and –ve rational numbers are represented on this line. What about the irrational numbers?

The fact is that all the irrational numbers are also represented by points of the line. Therefore, we postulate: -

Postulate: A (1 – 1) correspondence can be established between the points of a line ℓ and the real numbers in such a way that:-

i) The number 0 corresponds to a point O of the line.

ii) The number 1 corresponds to a point A of the line.

iii) If x_1, x_2 are the numbers corresponding to two points P_1, P_2, then the distance between P_1 and P_2 will be $|x_1 - x_2|$.

It is evident that the above correspondence will be such that corresponding to any real number there will be one and only one point on the line and vice versa.

When a (1 – 1) correspondence between the points of a line $x'x$ and the real numbers has been established in the manner described above, the line is called the **real line** and the real number, say x, corresponding to any point P of the line is called the **coordinate** of the point.

1.5.1 The Real Plane or The Coordinate Plane

We know that the *cartesian product* of two non-empty sets A and B, denoted by $A \times B$, is the set: $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

The members of a *cartesian* product are **ordered pairs.**

The cartesian product $\Re\times\Re$ where $\Re$ is the set of real numbers is called the **cartesian plane**.

By taking two perpendicular lines $x'ox$ and $y'oy$ as coordinate axes on a geometrical plane and choosing a convenient unit of distance, elements of $\Re\times\Re$ can be represented on the plane in such a way that there is *a* (1–1) correspondence between the elements of $\Re\times\Re$ and points of the plane.

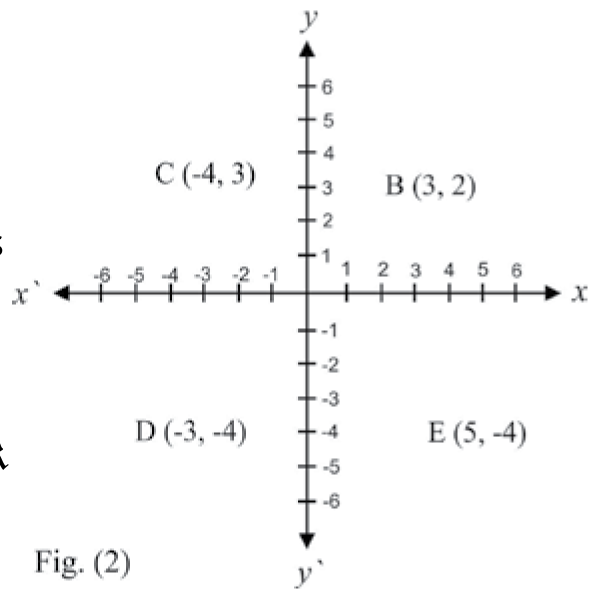

Fig. (2)

The geometrical plane on which coordinate system has been specified is called the **real plane** or the **coordinate plane**.

Ordinarily we do not distinguish between the Cartesian plane $\Re\times\Re$ and the coordinate plane whose points correspond to or represent the elements of $\Re\times\Re$.

If a point *A* of the coordinate plane corresponds to the ordered pair (*a*, *b*) then *a*, *b* are called the **coordinates** of *A*. *a* is called the *x* - coordinate or **abscissa** and *b* is called the y - coordinate or **ordinate**.

In the figure shown above, the coordinates of the points*B, C, D* and *E* are (3, 2), (–4, 3), (–3, –4) and (5, –4) respectively.

Corresponding to every ordered pair (*a, b*) $\in\Re\times\Re$ there is one and only one point in the plane and corresponding to every point in the plane there is one and only one ordered pair (*a, b*) in $\Re\times\Re$.

There is thus a (1 – 1) correspondence between $\Re\times\Re$ and the plane.

1.6 Geometrical Representation of Complex Numbers The Complex Plane

We have seen that there is a (1–1) correspondence between the elements (ordered pairs) of the Cartesian plane $\Re\times\Re$ and the complex numbers. Therefore, there is a (1– 1) correspondence between the points of the coordinate plane and the complex numbers. We can, therefore, represent complex numbers by points of the coordinate plane. In this representation every complex number will be represented by one and only one point of

the coordinate plane and every point of the plane will represent one and only one complex number. The components of the complex number will be the coordinates of the point representing it. In this representation the ***x*-axis** is called the real axis and the ***y*-axis** is called the **imaginary axis**. The coordinate plane itself is called the **complex plane** or **z – plane**.

By way of illustration a number of complex numbers have been shown in figure 3.

The figure representing one or more complex numbers on the *complex plane* is called an **Argand diagram**. Points on the **x-axis** represent **real numbers** whereas the points on the **y-axis** represent **imaginary numbers.**

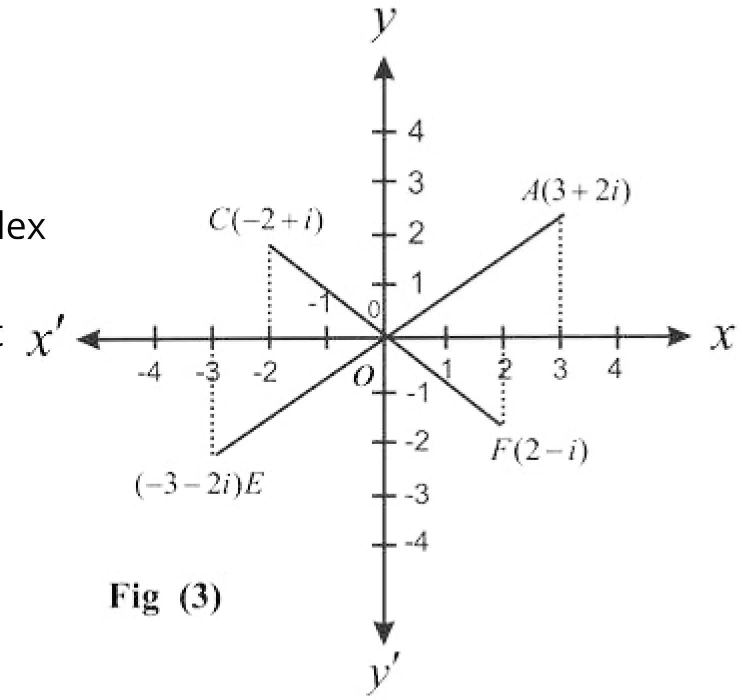

Fig (3)

In fig (4), *x*, *y* are the coordinates of a point.

It represents the complex number *x* + *iy*.

The *real number* $\sqrt{x^2+y^2}$ is called **the modulus of the complex number *a* + *ib*.**

In the figure $\overline{MA}\perp\overrightarrow{o\,x}$

$\therefore \overline{OM}=x,\ \overline{MA}=y$

In the right-angled triangle *OMA*, we have, by Pythagoras theorem,

$$\left|\overline{OA}\right|^2=\left|\overline{OM}\right|^2+\left|\overline{MA}\right|^2$$

$$\therefore \left|\overline{OA}\right|=\sqrt{x^2+y^2}$$

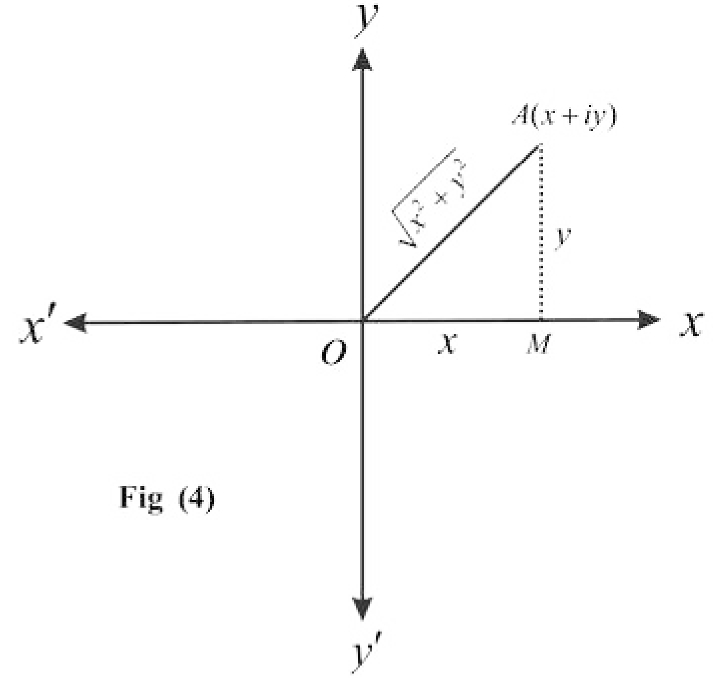

Fig (4)

Thus $\left|\overline{OA}\right|$ represents the modulus of *x* + *iy*. In other words: **The modulus of a complex number is the distance from the origin of the point representing the number.**

The modulus of a complex number is generally denoted as: $|x+iy|$ or $|(x,y)|$. For convenience, a complex number is denoted by z.

If $z = x + iy = (x, y)$, then

$$|z| = \sqrt{x^2+y^2}$$

Example 1: Find moduli of the following complex numbers :

(i) $1-i\sqrt{3}$ (ii) 3 (iii) $-5i$ (iv) $3+4i$

Solution:

i) Let $z = 1-i\sqrt{3}$

or $z = 1+i\left(-\sqrt{3}\right)$

$\therefore |z| = \sqrt{(1)^2+(-\sqrt{3})^2}$

$= \sqrt{1+3} = 2$

ii) Let $z = 3$

or $z = 3 + 0.i$

$\therefore |z| = \sqrt{(3)^2+(0)^2} = 3$

iii) Let $z = -5i$

or $z = 0 + (-5)i$

$\therefore |z| = \sqrt{0^2+(-5)^2} = 5$

iv) Let $z = 3 + 4i$

$\therefore |z| = \sqrt{(3)^2+(4)^2}$

Theorems: $\forall z, z_1, z_2 \in C$,

i) $|-z| = |z| = |\bar{z}| = |-\bar{z}|$

ii) $\bar{\bar{z}} = z$

iii) $z\bar{z} = |z|^2$

iv) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

vi) $|z_1.z_2| = |z_1|.|z_2|$

Proof :(i): Let $z = a+ib$,

So, $-z = -a-ib, \bar{z} = a-ib$ and $-\bar{z} = -a+ib$

$$\therefore |-z| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2} \quad (1)$$

$$|z| = \sqrt{a^2+b^2} \quad (2)$$

$$|\bar{z}| = \sqrt{(a)^2+(b)^2} = \sqrt{a^2+b^2} \quad (3)$$

$$|-\bar{z}| = \sqrt{(-a)^2+(b)^2} = \sqrt{a^2+b^2} \quad (4)$$

By equations (1), (2), (3) and (4) we conclude that

$$|-z| = |z| = |\bar{z}| = |-\bar{z}|$$

(ii) Let $z = a + ib$

So that $\bar{z} = a-ib$

Taking conjugate again of both sides, we have

$$\bar{\bar{z}} = a+ib = z$$

(iii) Let $z = a + ib$ so that $\bar{z} = a-ib$

$$\begin{aligned}\therefore z.\bar{z} &= (a+ib)(a-ib)\\ &= a^2 - iab + iab - i^2b^2\\ &= a^2-(-1)b^2\\ &= a^2+b^2 = |z|^2\end{aligned}$$

(iv) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$z_1 + z_2 = (a + ib) + (c + id)$

$= (a + c) + i(b + d)$

so, $\overline{z_1+z_2} = \overline{(a+c)+i(b+d)}$ (Taking conjugate on both sides)

$= (a + c) - i(b + d)$

$= (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2$

(v) Let $z_1 = a + ib$ and $z_2 = c + id$, where $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

$$= \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \qquad \text{(Note this step)}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} \qquad (1)$$

Now $\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{\overline{a+ib}}{\overline{c+id}} = \dfrac{a-ib}{c-id}$

$$= \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$$

$$= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} \qquad (2)$$

From (1) and (2), we have

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

(vi) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$|z_1.z_2| = |(a+ib)(c+id)|$$

$$= |(ac-bd)+(ad+bc)i|$$

$$= \sqrt{(ac-bd)^2+(ad+bc)^2}$$

$$= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$= |z_1|.|z_2|$$

This result may be stated thus: -

The modulus of the product of two complex numbers is equal to the product of their moduli.

(vii) Algebraic proof of this part is tedius. Therefore, we prove it geometrically.

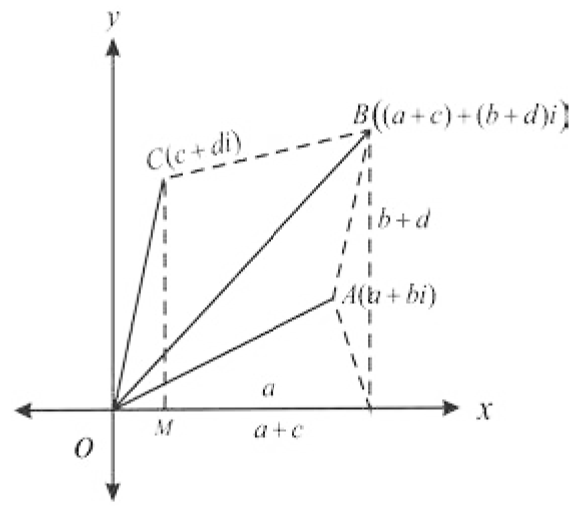

Fig (5)

In the figure point *A* represents $z_1 = a + ib$ and point *C* represents $z_2 = c + id$. We complete the parallelogram *OABC*. From the figure, it is evident that coordinates of *B* are $(a + c, b + d)$, therefore, *B* represents

$z_1 + z_2 = (a + c) + (b + d)i$ and $|\overline{OB}| = |z_1 + z_2|$.

Also $|\overline{OA}| = |z_1|$, $\qquad |\overline{AB}| = |\overline{OC}| = |z_2|$.

In the $\triangle OAB$; $OA + AB > OB$ $\qquad$ ($OA = m\overline{OA}$ etc.)

$$\therefore |z_1| + |z_2| > |z_1 + z_2| \qquad (1)$$

Also in the same triangle, $OA - AB < OB$

$$\therefore |z_1| - |z_2| < |z_1 + z_2| \qquad (2)$$

Combining (1) and (2), we have

$$|z_1| - |z_2| < |z_1 + z_2| < |z_1| + |z_2| \qquad (3)$$

which gives the required results with inequality signs.

Results with equality signs will hold when the points A and C representing z_1 and z_2 become collinear with B. This will be so when $\frac{a}{b}=\frac{c}{d}$ (see fig (6)).

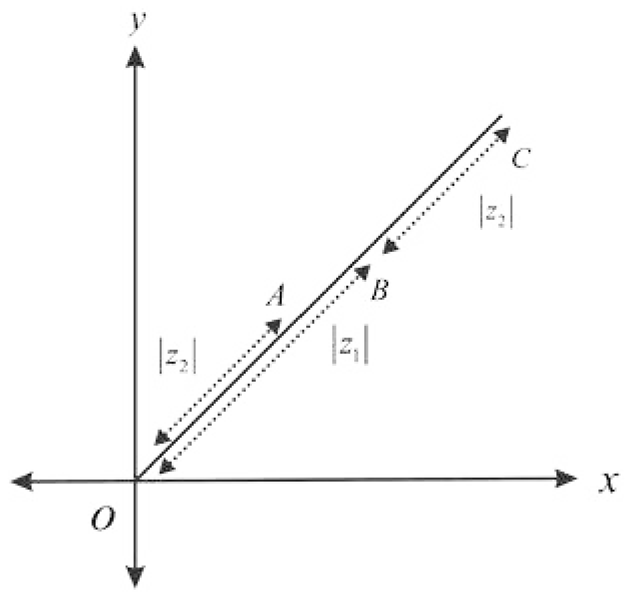

Fig (6)

In such a case $|z_1|+|z_2| = |\overline{OB}|+|\overline{OA}|$

$$=|\overline{OB}|+|\overline{BC}|$$
$$=|\overline{OC}|$$
$$=|z_1+z_2|$$

Thus $\quad |z_1+z_2| = |z_1|+|z_2|$

The second part of result (vii) namely

$$|z_1+z_2| \le |z_1|+|z_2|$$

is analogue of the triangular inequality*. In words, it may be stated thus: - The modulus of the sum of two complex numbers is less than or equal to the sum of the moduli of the numbers.

Example 2: If z_1 = 2+ i, z_2 = 3 – 2i, z_3 = 1 + 3i then express $\frac{\overline{z_1}\,\overline{z_3}}{z_2}$ in the form a + ib

(Conjugate of a complex number z is denoted as $\overline{z}$)

Solution:

$$\frac{\overline{z_1}\,\overline{z_3}}{z_2} = \frac{\overline{(2+i)}\,\overline{(1+3i)}}{3-2i} = \frac{(2-i)(1-3i)}{3-2i}$$

$$= \frac{(2-3)+(-6-1)i}{3-2i} = \frac{-1-7i}{3-2i}$$
$$= \frac{(-1-7i)(3+2i)}{(3-2i)(3+2i)}$$
$$= \frac{(-3+14)+(-2-21)i}{3^2+2^2} = \frac{11}{13} \quad \frac{23}{13}i$$

Example 3: Show that, $\forall\, z_1, z_2 \in C,\ \overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$

Solution: Let $z_1 = a + bi,\ z_2 = c + di$

$$\overline{z_1 z_2} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)(ad+bc)i}$$
$$=(ac-bd)-(ad+bc)i \qquad (1)$$

$$\overline{z_1}.\overline{z_2} = \overline{(a+bi)} = \overline{(c+di)}$$
$$=(a-bi)(c-di)$$
$$=(ac-bd)+(-ad-bc)i$$

$$\overline{z_1}.\overline{z_2} = \overline{(a+bi)}\,\overline{(c+di)}$$
$$=(a-bi)(c-di)$$
$$=(ac-bd)+(-ad-bc)i \qquad (2)$$

Thus from (1) and (2) we have, $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$

Polar form of a Complex number: Consider adjoining diagram representing the complex number $z = x + iy$. From the diagram, we see that $x = r\cos\theta$ and $y = r\sin\theta$ where $r = |z|$ and θ is called argument of z.

Hence $\quad x + iy = r\cos\theta + r\sin\theta \qquad$ ....(i)

where $r=\sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

Equation (i) is called the polar form of the complex number z.

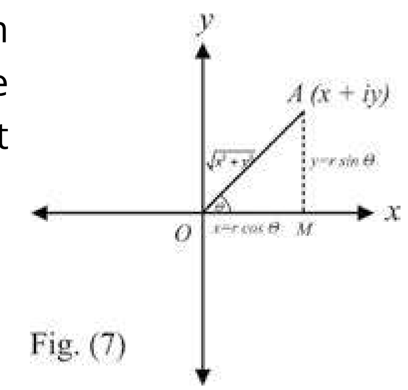

Fig. (7)

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*In any triangle the sum of the lengths of any two sides is greater than the length of the third side and difference of the lengths of any two sides is less than the length of the third side.

Example 4: Express the complex number $1+i\sqrt{3}$ in polar form.

Solution:

Step-I: Put $r\cos\theta = 1$ and $r\sin\theta = \sqrt{\ }$

Step-II: $r^2 = (1)^2 + (\sqrt{3})^2$

$\Rightarrow r^2 = 1+3 = 4 \quad \Rightarrow r = 2$

Step-III: $\theta = \tan^{-1}\dfrac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = 60^\circ$

Thus $1+i\sqrt{3} = 2\cos 60^\circ + i2\sin 60^\circ$

De Moivre's Theorem : -

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \ \forall n \in Z$

Proof of this theorem is beyond the scope of this book.

1.7 To find real and imaginary parts of

i) $(x + iy)^n$ **ii)** $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n, \ x_2+iy_2 \neq 0$

for $n = \pm1, \pm2, \pm3, \ldots$

i) Let $x = r\cos\theta$ and $y = r\sin\theta$, then

$(x + iy)^n = (r\cos\theta + ir\sin\theta)^n$

$= (r\cos\theta + ir\sin\theta)^n$

$= [r(\cos\theta + i\sin\theta)]^n$

$= r^n(\cos\theta + i\sin\theta)^n$

$= r^n(\cos n\theta + i\sin n\theta)$ (By De Moivre's Theorem)

$= r^n \cos n\theta + ir^n \sin n\theta$

Thus $r^n \cos n\theta$ and $r^n \sin n\theta$ are respectively the real and imaginary parts of $(x + iy)^n$.

Where $r = \sqrt{x^2 \quad y^2}$ and $\theta \quad tan^{-1}\dfrac{x}{y}$.

ii) Let $x_1 + iy_1 = r_1\cos\theta_1 + r_1\sin n\theta_1$ and $x_2 + iy_2 = r_2\cos\theta_2 + r_2\sin n\theta_2$ then,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{r_1\cos\theta_1 + r_1 i\sin\theta_1}{r_2\cos\theta_2 + r_2 i\sin\theta_2}\right)^n \quad \frac{r_1^n(\cos\theta_1 + i\sin\theta_1)^n}{r_2^n(\cos\theta_2 + i\sin\theta_2)^n}$$

$$= \frac{r_1^n}{r_2^n}(\cos\theta_1 \quad i\sin\theta_1)^n(\cos\theta_2 \quad i\sin\theta_2)^{-n}$$

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 + i\sin n\theta_1)\big(\cos(-n\theta)_2 + i\sin(-n\theta_2)\big),$$

(By De Moivre's Theorem)

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2), \ (\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta)$$

$$= \frac{r_1^n}{r_2^n}, \big[(\cos n\theta_1 \cos n\theta_2 \quad \sin n\theta_1 \sin n\theta_2)$$

$$+ i(\sin n\theta_1 \cos n\theta_2 - \cos n\theta_1 \sin n\theta_2)\big]$$

$$= \frac{r_1^n}{r_2^n}\big[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\big] \because \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\text{and } \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{r_1^n}{r_2^n}\big[\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\big]$$

$$= \frac{r_1^n}{r_2^n}\big[\cos n(\theta_1 \quad \theta_2) + i\sin n(\theta_1 - \theta_2)\big]$$

Thus $\dfrac{r_1^n}{r_2^n}\cos n(\theta_1 - \theta_2)$ and $\dfrac{r_1^n}{r_2^n}\sin n(\theta_1 - \theta_2)$ are respectively the real and imaginary parts of $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n, \ x_2+iy_2 \neq 0$

where $r_1 = \sqrt{x_1^2 + y_1^2}\,;\ \theta_1 = \tan^{-1}\frac{y_1}{x_1}$ and $r_2 = \sqrt{x_2^2 \quad y_2^2}\,;\ \theta_2 = \tan^{-1}\frac{y_2}{x_2}$

Example 5: Find out real and imaginary parts of each of the following complex numbers.

i) $\left(\sqrt{3}+\text{i}\right)^3$ ii) $\left(\dfrac{1-\sqrt{3}\,\text{i}}{1+\sqrt{3}\,\text{i}}\right)^5$

Solution:

i) Let $r\cos\theta=\sqrt{3}$ and $r\sin\theta=1$ where

$$r^2=(\sqrt{3})^2+1^2 \text{ or } r=\sqrt{3+1}=2 \text{ and } \theta=\tan^{-1}\frac{1}{\sqrt{3}}=30^o$$

So, $\left(\sqrt{3}+i\right)^3=(r\cos\theta+ir\sin\theta)^3$

$=r^3(\cos3\theta+i\sin3\theta)$ (By De Moivre's Theorem)

$=2^3(\cos90^\circ+i\sin90^\circ)$

$=8(0+i.1)$

$=8i$

Thus 0 and 8 are respectively real and imaginary Parts of $\left(\sqrt{3}+\text{i}\right)^3$.

ii) Let $r_1\cos\theta_1=1$ and $r_1\sin\theta_1=-\sqrt{3}$

$$\Rightarrow r_1=\sqrt{(1)^2+(-\sqrt{3})^2}=\sqrt{1+3}=2 \text{ and } \theta_1=\tan^{-1}\ -\frac{\sqrt{3}}{1}\ -60^o$$

Also Let $r_2\cos\theta_2=1$ and $r_2\sin\theta_2=\sqrt{3}$

$$\Rightarrow r_2=\sqrt{(1)^2+(\sqrt{3})^2}=\sqrt{1+3}=2 \text{ and } \theta_2\ \tan^{-1}\frac{\sqrt{3}}{1}=60^o$$

So, $\left(\dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5=\left[\dfrac{2\left(\cos(-60^o)+i\sin(-60^O)\right)}{2\left(\cos(60^o)+i\sin(60^O)\right)}\right]^5$

$$=\frac{\left(\cos(-60^\circ)+i\sin(-60^\circ)\right)^5}{\left(\cos(60^\circ)+i\sin(60^\circ)\right)^5}$$

$$=\left(\cos(-60^o)+i\sin(-60^O)\right)^5\left(\cos(60^o)+i\sin(60^O)\right)^{-5}$$

$$=\left(\cos(-300^o)+i\sin(-300^o)\right)\left(\cos(-300^o)+i\sin(-300^o)\right)$$

$$=\left(\cos(300^o)\quad i\sin(300^o)\right)\left(\cos(300^o)\quad i\sin(300^o)\right)-\because\cos(\quad\theta)=\cos\theta$$
$$\text{and }\sin(-\theta)=-\sin\theta$$

$$=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$$

Thus $\dfrac{-1}{2}$, $\dfrac{\sqrt{3}}{2}$ are respectively real and imaginary parts of $\left(\dfrac{1-\sqrt{3}\,\text{i}}{1+\sqrt{3}\,\text{i}}\right)^5$

Exercise 1.3

1. Graph the following numbers on the complex plane: -

i) $2+3i$ ii) $2-3i$ iii) $-2-3i$ iv) $-2+3i$

v) -6 vi) i vii) $\frac{3}{5}-\frac{4}{5}i$ viii) $-5-6i$

2. Find the multiplicative inverse of each of the following numbers: -

i) $-3i$ ii) $1-2i$ iii) $-3-5i$ iv) (1, 2)

3. Simplify

i) i^{101} ii) $(-ai)^4$, $a\in\Re$ iii) i^{-3} iv) i^{-10}

4. Prove that $\bar{z}=z$ iff z is real.

5. Simplify by expressing in the form *a* + *bi*

i) $5+2\sqrt{-4}$ ii) $(2+\sqrt{-3})(3+\sqrt{-3})$

iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$ iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

6. Show that $\forall z\in C$

i) $z^2-\bar{z}^2$ is a real number. ii) $(z-\bar{z})^2$ is a real number.

7. Simplify the following

i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$ ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

iiii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$ iv) $(a+bi)^2$

v) $(a+bi)^{-2}$ vi) $(a+bi)^3$

vii) $(a-bi)^3$ viii) $(3-\sqrt{-4})^{-3}$

CHAPTER

2

Sets Functions and Groups

Animation 2.1: Function
Source & Credit: elearn.punjab

2.1 Introduction

We are familiar with the notion of a **set** since the word is frequently used in everyday speech, for instance, water set, tea set, sofa set. It is a wonder that mathematicians have developed this ordinary word into a mathematical concept as much as it has become a language which is employed in most branches of modern mathematics.

For the purposes of mathematics, a set is generally described as a **well-defined collection of distinct objects.** By a *well-defined collection* is meant a collection, which is such that, given any object, we may be able to decide whether the object belongs to the collection or not. By *distinct objects* we mean objects no two of which are identical (same).

The objects in a set are called its **members** or **elements**. Capital letters *A, B, C, X, Y, Z* etc., are generally used as names of sets and small letters *a, b, c, x, y, z* etc., are used as *members* of sets.

There are three different ways of describing a set

i) **The Descriptive Method:** A set may be described in words. For instance, the set of all vowels of the English alphabets.

ii) **The Tabular Method:** A set may be described by listing its elements within brackets. If *A* is the set mentioned above, then we may write:

$$A = \{a,e,i,o,u\}.$$

iii) **Set-builder method:** It is sometimes more convenient or useful to employ the method of set-builder notation in specifying sets. This is done by using a symbol or letter for an arbitrary member of the set and stating the property common to all the members.

Thus the above set may be written as:

$A = \{ x \mid x$ is a vowel of the English alphabet$\}$

This is read as *A* is the set of all x such that x is a vowel of the English alphabet.

The symbol used for **membership** of a set is $\in$. Thus $a \in A$ means ***a* is an element of *A*** or ***a* belongs to *A***. $c \notin A$ means ***c* does not belong to *A*** or ***c* is not a member of *A***. Elements of a set can be anything: people, countries, rivers, objects of our thought. In algebra we usually deal with sets of numbers. Such sets, alongwith their names are given below:-

N = The set of all natural numbers = $\{1,2,3,\ldots\}$

W = The set of all whole numbers = $\{0,1,2,\ldots\}$

Z = The set of all integers = $\{0,\pm1,+2\ldots.\}$.

Z' = The set of all negative integers = $\{-1,-2,-3,\ldots\}$.

O = The set of all odd integers = $\{\pm1,\pm3,\pm5,\ldots\}$.

E = The set of all even integers = $\{0,\pm2,\pm4,\ldots\}$.

Q = The set of all rational numbers = $\left\{x \,\middle|\, x = \frac{p}{q} \text{ where p,q} \in Z \text{ and q} \neq 0\right\}$

Q' = The set of all irrational numbers = $\left\{x \,\middle|\, x \neq \frac{p}{q} \text{ where p,q} \in Z \text{ and q} \neq 0\right\}$

$\mathbb{R}$ = The set of all real numbers = $Q \cup Q'$

Equal Sets: Two sets *A* and *B* are equal i.e., *A*=*B*, if and only if they have the same elements that is, if and only if every element of each set is an element of the other set.

Thus the sets { 1, 2, 3 } and { 2, 1, 3} are equal. From the definition of equality of sets it follows that a mere change in the order of the elements of a set does not alter the set. In other words, while describing a set in the tabular form its elements may be written in any order.

Note: (1) *A* = *B* if and only if they have the same elements means if *A* = *B* they have the same elements and if *A* and *B* have the same elements then *A* = *B*.

(2) The phrase if and only if is shortly written as "iff ".

Equivalent Sets: If the elements of two sets *A* and *B* can be paired in such a way that each element of *A* is paired with one and only one element of *B* and vice versa, then such a pairing is called a one-to-one correspondence between *A* and *B* e.g., if *A* = {Billal, Ahsan, Jehanzeb} and *B* = {Fatima, Ummara, Samina} then six different (1 - 1) correspondences can be established between *A* and *B*

Two of these correspondences are shown below; -

i). {Billal, Ahsan, Jehanzeb }
$\updownarrow$ $\updownarrow$ $\updownarrow$
{Fatima, Ummara, Samina }

ii). {Billal, Ahsan, Jehanzeb)

$\updownarrow \quad \updownarrow \quad \updownarrow$

{Fatima, Samina, Ummara)

(Write down the remaining 4 correspondences yourselves)

Two sets are said to be equivalent if *a* (1 – 1) correspondence can be established between them In the above example *A* and *B* are equivalent sets.

Example 1: Consider the sets *N*= {1, 2, 3,... } and *O* = {1, 3, 5,...}

We may establish (1–1) correspondence between them in the following manner:

$$\begin{array}{cccccc} \{1, & 2, & 3, & 4, & 5, & ...\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{1, & 3, & 5, & 7, & 9, & ...\} \end{array}$$

Thus the sets *N* and *O* are equivalent. But notice that they are not equal.

Remember that two equal sets are necessarily equivalent, but the converse may not be true i.e., two equivalent sets are not necessarily equal.

Sometimes, the symbol ~ is used to mean **is equivalent to**. Thus *N*~*O*.

Order of a Set: There is no restriction on the number of members of a set. A set may have 0, 1, 2, 3 or any number of elements. Sets with zero or one element deserve special attention. According to the everyday use of the word set or collection it must have at least two elements. But in mathematics it is found convenient and useful to consider sets which have only one element or no element at all.

A set having only one element is called a **singleton set** and a set with no element (zero number of elements) is called the **empty set** or **null set**.The empty set is denoted by the symbol ϕ or { }.The set of odd integers between 2 and 4 is a singleton i.e., the set {3} and the set of even integers between the same numbers is the empty set.

The solution set of the equation $x^2 + 1 = 0$, in the set of real numbers is also the empty set. Clearly the set {0} is a *singleton set* having zero as its only element, and not the empty set.

Finite and Infinite sets: If a set is equivalent to the set {1, 2, 3,...*n*} for some fixed natural number *n*, then the set is said to be *finite* otherwise *infinite*.

Sets of number *N, Z, Z'*etc., mentioned earlier are infinite sets.

The set {1, 3, 5,.......9999} is a finite set but the set { 1, 3, 5, ...}, which is the set of all positive odd natural numbers is an infinite set.

Subset: If every element of a set *A* is an element of set *B*, then *A* is a *subset* of *B*. Symbolically this is written as: $A \subseteq B$ (*A* is subset of *B*)

In such a case we say *B* is a super set of *A*. Symbolically this is written as:

$B \supseteq A$ {*B* is a superset of *A*)

Note: The above definition may also be stated as follows:

$A \subseteq B$ iff $x \in A \Rightarrow x \in B$

Proper Subset: If *A* is a subset of *B* and *B* contains at least one element which is not an element of *A*, then A is said to be a *proper subset* of *B*. In such a case we write: $A \subset B$ (*A* is a proper subset of *B*).

Improper Subset: If *A* is subset of *B* and *A* = *B*, then we say that *A* is an improper subset of *B*. From this definition it also follows that every set *A* is an improper subset of itself.

Example 2: Let *A* = { *a*, *b*, *c* }, *B* = { *c*, *a*, *b*} and *C* = { *a*, *b*, *c*, *d*}, then clearly

$A \subset C$, $B \subset C$ but $A = B$ and $B = A$.

Notice that each of *A* and *B* is an improper subset of the other because *A* = *B*

Note: When we do not want to distinguish between proper and improper subsets, we may use the symbol $\subseteq$ for the relationship. It is easy to see that: $N \subset Z \subset Q \subset \mathbb{R}$.

Theorem 1.1: The empty set is a subset of every set.

We can convince ourselves about the fact by rewording the definition of subset as follows: -

A is subset of *B* if it contains no element which is not an element of *B*.

Obviously an empty set does not contain such element, which is not contained by another set.

Power Set: A set may contain elements, which are sets themselves. For example if: *C* = Set of classes of a certain school, then elements of *C* are sets themselves because each class is a set of students. An important set of sets is the *power set* of a given set.

The *power set* of a set *S* denoted by *P* (*S*) is the set containing all the possible subsets of *S*.

Example 3: If $A = \{a, b\}$, then $P(A)=\{\Phi, \{a\}, \{b\}, \{a,b\}\}$
Recall that the empty set is a subset of every set and every set is its own subset.

Example 4: If $B = \{1, 2, 3\}$, then

$$P(B) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Example 5: If $C = \{a, b, c, d\}$, then

Example 6: If $D = \{a\}$, then $P(D)=\{\Phi,\{a\}\}$

Example 7: If $E = \{\ \}$, then $P(E)=\{\Phi\}$

Note: (1) The power set of the empty set is not empty.
(2) Let *n* (*S*) denoted the number of elements of a set *S*, then *n*{*P(S)*} denotes the number of elements of the power set of *S*. From examples 3 to 7 we get the following table of results:

n(s)	0	1	2	3	4	5
n{p(s)}	$1=2^0$	$2=2^1$	$4=2^2$	$8=2^3$	$16=2^4$	$32=2^5$

In general if $n(S) = m$, then, $n\,P(S) = 2^m$

Universal Set: When we are studying any branch of mathematics the sets with which we have to deal, are generally subsets of a bigger set. Such a set is called the **Universal set** or the **Universe of Discourse**. At the elementary level when we are studying arithmetic, we have to deal with *whole numbers* only. At that stage the set of whole numbers can be treated as *Universal Set*. At a later stage, when we have to deal with negative numbers also and fractions, the set of the *rational numbers* can be treated as the *Universal Set*.

For illustrating certain concepts of the Set Theory, we sometimes consider quite small sets (sets having small number of elements) to be universal. This is only an academic artificiality.

Exercise 2.1

1. Write the following sets in set builder notation:
   i) $\{1, 2, 3, ...,1000\}$ ii) $\{0, 1, 2, ..., 100\}$
   iii) $\{0, \pm 1, \pm 2, ... \pm 1000\}$ iv) $\{0, -1, -2, ...,-500\}$
   v) $\{100, 101, 102, ...,400\}$ vi) $\{-100, -101, -102, ..., -500\}$
   vii) {Peshawar, Lahore, Karachi, Quetta}
   viii) { January, June, July }
   xi) The set of all odd natural numbers
   x) The set of all rational numbers
   xi) The set of all real numbers between 1and 2,
   xii) The set of all integers between – 100 and 1000

2. Write each of the following sets in the descriptive and tabular forms:-
   i) $\{x \mid x \in N \wedge x \leq 10\}$ ii) $\{x \mid x \in N \wedge 4 < x < 12\}$
   iii) $\{x \mid x \in Z \wedge -5 < x < 5\}$ iv) $\{x \mid x \in E \wedge 2 < x \leq 4\}$
   v) $\{x \mid x \in P \wedge x < 12\}$ vi) $\{x \mid x \in O \wedge 3 < x < 12\}$
   vii) $\{x \mid x \in E \wedge 4 \leq x \leq 10\}$ viii) $\{x \mid x \in E \wedge 4 < x < 6\}$
   ix) $\{x \mid x \in O \wedge 5 \leq x \leq 7\}$ x) $\{x \mid x \in O \wedge 5 \leq x < 7\}$
   xi) $\{x \mid x \in N \wedge x + 4 = 0\}$ xi) $\{x \mid x \in Q \wedge x^2 = 2\}$
   xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$ xiv) $\{x \mid x \in Q \wedge x = -x\}$
   xv) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$ xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin Q\}$
3. Which of the following sets are finite and which of these are infinite?
   i) The set of students of your class.
   ii) The set of all schools in Pakistan.
   iii) The set of natural numbers between 3 and 10.
   iv) The set of rational numbers between 3 and 10.
   v) The set of real numbers between 0 and 1.
   vi) The set of rationales between 0 and 1.
   vii) The set of whole numbers between 0 and 1
   viii) The set of all leaves of trees in Pakistan.

ix) $P(N)$ x) $P\{a, b, c\}$
xi) $\{1,2,3,4,\ldots\}$ xii) $\{1,2,3,\ldots,100000000\}$
xiii) $\{x\ x | x \in \mathbb{R} \wedge x \neq x\}$ xiv) $\{x | x \in \mathbb{R} \wedge x^2 = -16\}$
xv) $\{x | x \in Q \wedge x^2 = 5\}$ xvi) $\{x | x \in Q \wedge 0 \leq x \leq 1\}$

4. Write two proper subsets of each of the following sets: -
i) $\{a, b, c\}$ ii) $\{0, 1\}$ iii) N iv) Z
v) Q vi) $\mathbb{R}$ vii) W viii) $\{x | x \in Q \wedge 0 < x \leq 2\}$

5. Is there any set which has no proper sub set? If so name that set.

6. What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$?

7. Which of the following sentences are true and which of them are false?
i) $\{1,2\} = \{2,1\}$ ii) $\Phi \subseteq \{\{a\}\}$ iii) $\{a\} \subseteq \{\{a\}\}$
v) $\{a\} \in \{\{a\}\}$ vi) $a \in \{\{a\}\}$ vii) $\Phi \in \{\{a\}\}$

8. What is the number of elements of the power set of each of the following sets?
i) $\{\}$ ii) $\{0,1\}$ iii) $\{1,2,3,4,5,6,7\}$
v) $\{0,1,2,3,4,5,6,7\}$ vi) $\{a, \{b, c\}\}$ vii) $\{\{a,b\},\{b,c\},\{d,e\}\}$

9. Write down the power set of each of the following sets: -
i) $\{9,11\}$ ii) $\{+, -, \times, \div\}$ iii) $\{\Phi\}$ iv) $\{a, \{b,c\}\}$

10. Which pairs of sets are equivalent? Which of them are also equal?
i) $\{a, b, c\}$, $\{1, 2, 3\}$
ii) The set of the first 10 whole members, $\{0, 1, 2, 3,\ldots,9\}$
iii) Set of angles of a quadrilateral *ABCD*,
set of the sides of the same quadrilateral.
iv) Set of the sides of a hexagon *ABCDEF*,
set of the angles of the same hexagon;

v) $\{1,2,3,4,\ldots\}$, $\{2,4,6,8,\ldots\}$ vi) $\{1,2,3,4,\ldots\}$, $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

viii) $\{5, 10, 15,\ldots,55555\}$, $\{5, 10, 15, 20,\ldots\}$

2.2 Operations on Sets

Just as operations of addition, subtraction etc., are performed on numbers, the operations of unions, intersection etc., are performed on sets. We are already familiar with them. A review of the main rules is given below: -

Union of two sets: The Union of two sets *A* and *B*, denoted by $A \cup B$, is the set of all elements, which belong to *A* or *B*. Symbolically;

$$A \cup B \ \{x | x \in A \vee x \in B\}$$

Thus if $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, then $A \cup B = \{1,2,3,4,5\}$

Notice that the elements common to *A* and *B*, namely the elements 2, 3 have been written only once in $A \cup B$ because repetition of an element of a set is not allowed to keep the elements distinct.

Intersection of two sets: The intersection of two sets *A* and *B*, denoted by $A \cap B$, is the set of all elements, which belong to both *A* and *B*. Symbolically;

$$\mathrm{A} \cap \mathrm{B} = \{x | x \in A \wedge x \in B\}$$

Thus for the above sets *A* and *B*, $A \cap B = \{2,3\}$

Disjoint Sets: If the intersection of two sets is the empty set then the sets are said to be disjoint sets. For example; if

S_1 = The set of odd natural numbers and S_2 = The set of even natural numbers, then S_1 and S_2 are disjoint sets.

The set of arts students and the set of science students of a college are disjoint sets.

Overlapping sets: If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets, e.g., if

$L = \{2,3,4,5,6\}$ and $M = \{5,6,7,8,9,10\}$, then *L* and *M* are two overlapping sets.

Complement of a set: The complement of a set *A*, denoted by A' or A^c relative to the universal set *U* is the set of all elements of *U*, which do not belong to *A*.

Symbolically: $A' = \{x | x \in U \wedge x \notin A\}$

For example, if *U*=*N*, then $E' = O$ and $O' = E$

Example 1: If *U* = set of alphabets of English language,
C = set of consonants,
W = set of vowels, then $C' = W$ and $W' = C$.

Difference of two Sets: The Difference set of two sets *A* and *B* denoted by *A*–*B* consists of all the elements which belong to *A* but do not belong to *B*.

The Difference set of two sets *B* and *A* denoted by *B*-*A* consists of all the elements, which belong to *B* but do not belong to *A*.

Symbolically, $A-B = \{x | x \in A \wedge x \notin B\}$ and $B-A = \{x | x \in B \wedge x \notin A\}$

Example 2: If *A* = {1,2,3,4,5}, *B* = {4,5,6,7,8,9,10}, then
A–*B* = {1,2,3} and *B*–*A* = {6,7,8,9,10}.

Notice that $A-B \neq B-A$.

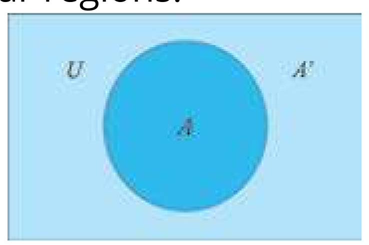
Note: In view of the definition of complement and difference set it is evident that for any set *A*, $A' = U - A$

2.3 Venn Diagrams

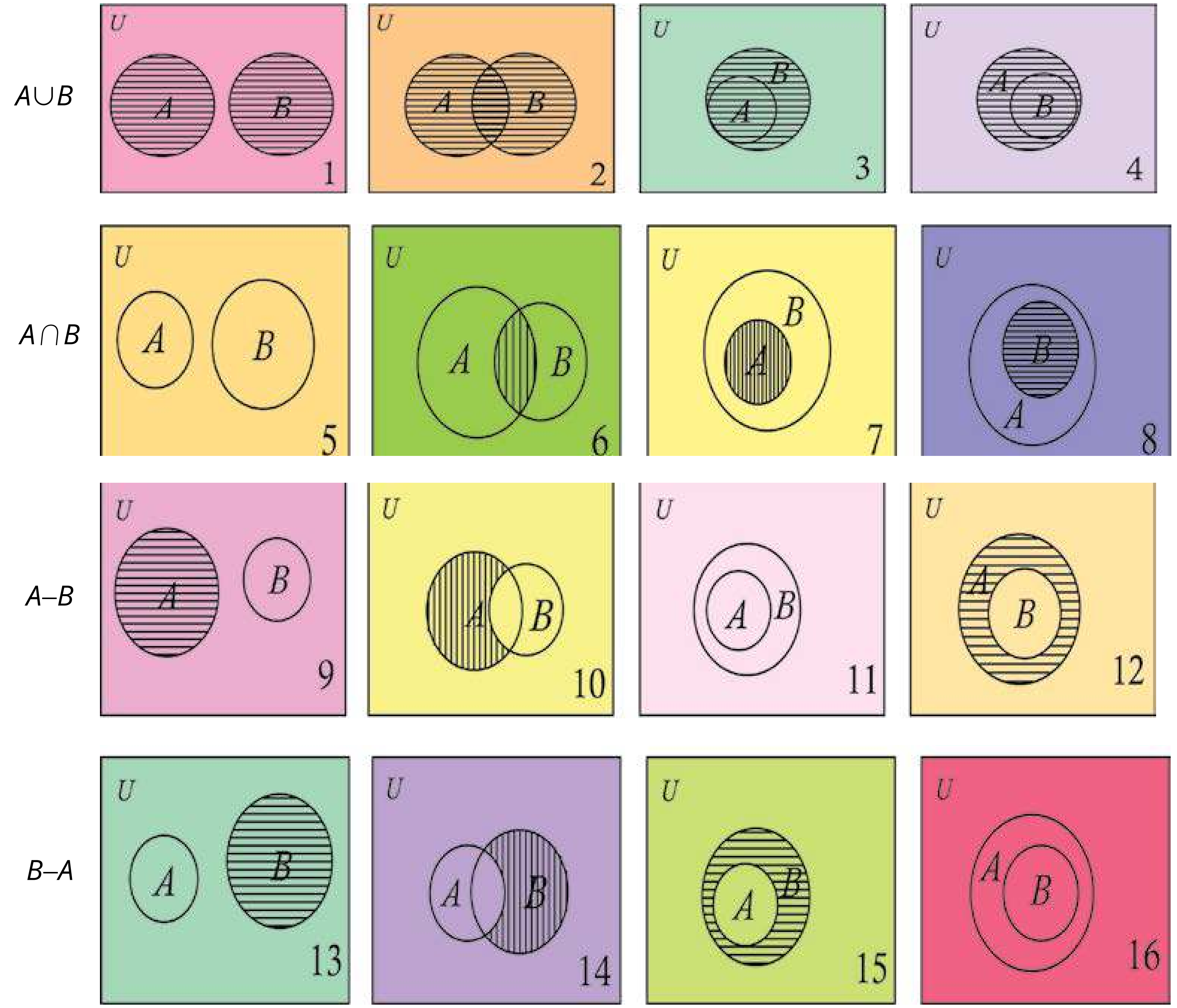
Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. They were first used by an English logician and mathematician John Venn (1834 to 1883 A.D).

In a Venn diagram, a rectangular region represents the universal set and regions bounded by simple closed curves represent other sets, which are subsets of the universal set. For the sake of beauty these regions are generally shown as circular regions.

In the adjoining figures, the shaded circular region represents a set *A* and the remaining portion of rectangle representing the universal set *U* represents *A*′ or *U* – *A*.



Below are given some more diagrams illustrating basic operations on two sets in different cases (lined region represents the result of the relevant operation in each case given below).



The above diagram suggests the following results: -

Fig No.	Relation between A and B	Result Suggested
1.	A and B disjoint sets $A \cap B = \Phi$	$A \cup B$ consists of all the elements of A and all the elements of B. Also $n(A \cup B) = n(A) + n(B)$
2.	A and B are overlapping $A \cap B \neq \Phi$	$A \cup B$ contains elements which are i) in A and not in B ii) in B and not in A iii) in both A and B. Also $n(A \cup B) = n(A) + n(B) - (A \cap B)$
3.	$A \subseteq B$	$A \cup B = B$; $n(A \cup B) = n(B)$
4.	$B \subseteq A$	$A \cup B = A$; $n(A \cup B) = n(A)$
5.	$A \cap B = \Phi$	$A \cap B = \Phi$; $n(A \cap B) = 0$
6.	$A \cap B \neq \Phi$	$A \cap B$ contains the elements which are in A *and* B
7.	$A \subseteq B$	$A \cap B = A$; $n(A \cap B) = n(A)$
8.	$B \subseteq A$	$A \cap B = B$; $n(A \cap B) = n(B)$
9.	A and B are disjoint sets.	$A - B = A$; $n(A - B) = n(A)$
10.	A and B are overlapping	$n(A - B) = n(A) - n\, A \cap B$
11.	$A \subseteq B$	$A - B = \Phi$; $n(A - B) = 0$
12.	$B \subseteq A$	$A - B \neq \Phi$; $n(A - B) = n(A) - n(B)$
13.	A and B are disjoint	$B - A = B$; $n(B - A) = n(B)$
14.	A and B are overlapping	$n(B - A) = n(B) - n(A \cap B)$
15.	$A \subseteq B$	$B - A \neq \Phi$; $n(B - A) = n(B) - n(A)$
16.	$B \subseteq A$	$B - A = \Phi$; $n(B - A) = 0$

Note (1) Since the empty set contains no elements, therefore, no portion of U represents it.

(2) If in the diagrams given on preceding page we replace B by the empty set (by imagining the region representing B to vanish).

$A \cup \Phi = A$ (FromFig. 1 o r 4)

$A \cap \Phi = \Phi$ (From Fig. 5 or 8)

$A - \Phi = A$ (From Fig. 9 or 12)

$\Phi - A = \Phi$ (FromFig. 13 or 16)

Also by replacing B by A (by imagining the regions represented by A and B to coincide), we obtain the following results:

$A \cup A = A$ (From fig. 3 or 4)

$A \cap A = A$ (From fig. 7 or 8)

$A - A = \Phi$ (From fig. 12)

Again by replacing B by U, we obtain the results: -

$A \cup U = U$ (From fig. 3); $A \cap U = A$ **(From fig. 7)**

$A - U = \Phi$ (From fig. 11); $U - A = A'$ **(From fig. 15)**

(3) Venn diagrams are useful only in case of abstract sets whose elements are not specified. It is not desirable to use them for concrete sets (Although this is erroneously done even in some foreign books).

Exercise 2.2

1. Exhibit $A \cup B$ and $A \cap B$ by Venn diagrams in the following cases: -

i) $A \subseteq B$ ii) $B \subseteq A$ iii) $A \cup A'$

iv) A and B are disjoint sets. v) A and B are overlapping sets

2. Show $A - B$ and $B - A$ by Venn diagrams when: -

i) A and B are overlapping sets ii) $A \subseteq B$ iii) $B \subseteq A$

3. Under what conditions on A and B are the following statements true?

i) $A \cup B = A$ ii) $A \cup B = B$ iii) $A - B = A$

iv) $A \cap B = B$ v) $n(A \cup B) = n(A) + n(B)$ vi) $n(A \cap B) = n(A)$

vii) $A - B = A$ vii) $n(A \cap B) = 0$ ix) $A \cup B = U$
x) $A \cup B = B \cup A$ xi) $n(A \cap B) = n(B)$. xii) $U - A = \Phi$

4. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5\}$ and $C = \{1,3,5,7,9\}$
List the members of each of the following sets: -
i) A^c ii) B^c iii) $A \cup B$ iv) $A - B$
v) $A \cap C$ vi) $A^c \cup C^c$ vii) $A^c \cup C$. viii) U^c

5. Using the Venn diagrams, if necessary, find the single sets equal to the following: -
i) A^c ii) $A \cap U$ iii) $A \cup U$ iv) $A \cup \Phi$ v) $\Phi \cap \Phi$

6. Use Venn diagrams to verify the following: -
i) $A - B = A \cap B^c$ ii) $(A - B)^c \cap B = B$

2.4 Operations on Three Sets

If A, B and C are three given sets, operations of union and intersection can be performed on them in the following ways: -
i) $A \cup (B \cup C)$ ii) $(A \cup B) \cup C$ iii) $A \cap (B \cup C)$
iv) $(A \cap B) \cap C$ v) $A \cup (B \cap C)$ vi) $(A \cap C) \cup (B \cap C)$
vii) $(A \cup B) \cap C$ viii) $(A \cap B) \cup C$. ix) $(A \cup C) \cap (B \cup C)$
Let $A = \{1, 2, 3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$
We find sets (i) to (iii) for the three sets (Find the remaining sets yourselves).
i) $B \cup C = \{2,3,4,5,6,7,8\}$, $A \cup (B \cup C) = \{1,2,3,4,5,6,7,8\}$
ii) $A \cup B = \{1,2,3,4,5\}$, $(A \cup B) \cup C = \{1,2,3,4,5,6,7,8\}$
iii) $B \cap C = \{3,4,5\}$, $A \cap (B \cap C) = \{3\}$

2.5 Properties of Union and Intersection

We now state the fundamental properties of union and intersection of two or three sets. Formal proofs of the last four are also being given.

Properties:
i) $A \cup B = B \cup A$ (Commutative property of Union)
ii) $A \cap B = B \cap A$ (Commutative property of Intersection)
iii) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative property of Union)
iv) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative property of Intersection).
v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributivity of Union over intersection)
vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity of intersection over Union)

vii) $(A \cup B)' = A' \cap B'$
viii) $(A \cap B)' = A' \cup B'$ De Morgan's Laws

Proofs of De Morgan's laws and distributive laws:

i) $(A \cup B)' = A' \cap B'$
Let $x \in (A \cup B)'$
$\Rightarrow x \notin A \cup B$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in A' \cap B'$
But x is an arbitrary member of $(A \cup B)'$ (1)
Therefore, (1) means that $(A \cup B)' \subseteq A' \cap B'$ (2)
Now suppose that $y \in A' \cap B'$
$\Rightarrow y \in A'$ and $y \in B'$
$\Rightarrow y \notin A$ and $y \notin B$
$\Rightarrow y \notin A \cup B$
$\Rightarrow y \in (A \cup B)'$
Thus $A' \cap B' \subseteq (A \cup B)'$
From (2) and (3) we conclude that (3)
$$(A \cup B)' = A' \cap B'$$

ii) $(A \cap B)' = A' \cup B'$
It may be proved similarly or deducted from (i) by complementation

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Let $x \in A \cup (B \cap C)$
$\Rightarrow x \in A$ or $x \in B \cap C$
$\Rightarrow$ If $x \in A$ it must belong to $A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad x \in (A\cup B)\cap(A\cup C)$

Also if $x \in B \cap C$, then $x \in B$ and $x \in C$. (1)

$\Rightarrow \quad x \in A\cup B$ and $x \in A\cup C$

$\Rightarrow \quad x \in (A\cup B)\cap(A\cup C)$

Thus $A\cup(B\cap C) \subseteq (A\cup B)\cap(A\cup C)$ (2)

Conversely, suppose that

$y \in (A\cup B)\cap(A\cup C)$

There are two cases to consider: -

$y \in A,\ y \notin A$

In the first case $y \in A\cup(B\cap C)$

If $y \notin A$, it must belong to B as well as C

i.e., $y \in (B\cap C)$

$\therefore y \in A\cup(B\cap C)$

So in either case

$y \in (A\cup B)\cap(A\cup C) \Rightarrow y \in A\cup(B\cap C)$

thus $(A\cup B)\cap(A\cup C) \subseteq A\cup(B\cap C)$ (3)

From (2) and (3) it follows that

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

iv) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

It may be proved similarly or deducted from (iii) by complementation

Verification of the properties:

Example 1: Let $A = \{1,2,3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$

i) $A\cup B = \{1,2,3\} \cup \{2,3,4,5\}$ $\quad B\cup A = \{2,3,4,5\} \cup \{1,2,3\}$

$= \{1,2,3,4,5\}$ $\quad = \{2,3,4,5,1\}$

$\therefore A\cup B = B\cup A$

ii) $A\cap B = \{1,2,3\}\cap\{2,3,4,5\}$ $\quad B\cap A = \{2,3,4,5\}\cap\{1,2,3\}$

$= \{2,3\}$ $\quad = \{2,3\}$

$\therefore A\cap B = B\cap A$

(iii) and (iv) Verify yourselves.

(v) $A\cup(B\cap C) = \{1,2,3\} \cup (\{2,3,4,5\}\cap\{3,4,5,6,7,8\})$

$= \{1,2,3\} \cup \{3,4,5\}$

$= \{1,2,3,4,5\}$ (1)

$(A\cup B)\cap(A\cup C) = (\{1,2,3\}\cup\{2,3,4,5\})\cap(\{1,2,3\}\cup\{3,4,5,6,7,8\})$

$= \{1,2,3,4,5\} \cap \{1,2,3,4,5,6,7,8\}$

$= \{1,2,3,4,5\}$ (2)

From (1) and (2),

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

vi) Verify yourselves.

vii) Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A\cup B = \{1,2,3\} \cup \{2,3,4,5\} = \{1,2,3,4,5\}$

$(A\cup B)' = \{6,7,8,9,10\}$ (1)

$A' = U - A = (4,5,6,7,8,9,10)$

$B' = U - B = \{1,6,7,8,9,10\}$

$A' \cap B' = (4,5,6,7,8,9,10) \cap \{1,6,7,8,9,10\}$

$= \{6,7,8,9,10\}$ (2)

From (1) and (2),

$(A\cup B)' = A' \cap B'$

viii) Verify yourselves.

Verification of the properties with the help of Venn diagrams.

i) and (ii): Verification is very simple, therefore, do it yourselves,

iii): In fig. (1) set A is represented by vertically lined region and $B\cup C$ is represented by horizontally lined region. The set $A\cup(B\cup C)$ is represented by the region which is lined either in one or both ways.

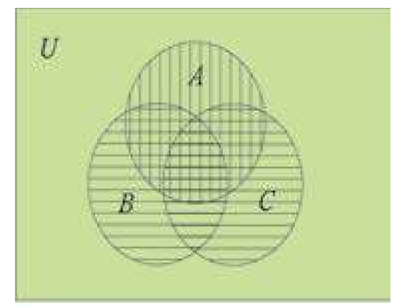


Fig (1)

A |||
$B\cup C$ ≡ ∨ #
$A\cup(B\cup C)$ |||, ≡ ∨ #

In figure(2) $A\cup B$ is represented by horizontally lined region and C by vertically lined region. $(A\cup B)\cup C$ is represented by the region which is lined in either one or both ways.

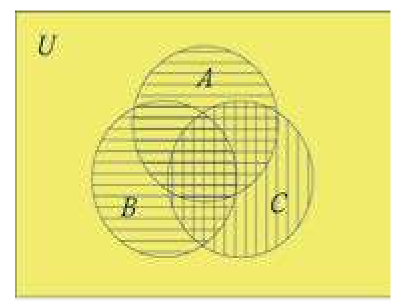


Fig (2)

$A\cup B$
C |||
$(A\cup B)\cup C$ ≡ ||| ∨ #

From fig (1) and (2) we can see that
$A \cup (B \cup C) = (A \cup B) \cup C$

(iv) In fig (3) doubly lined region represents.
$A \cap (B \cap C)$

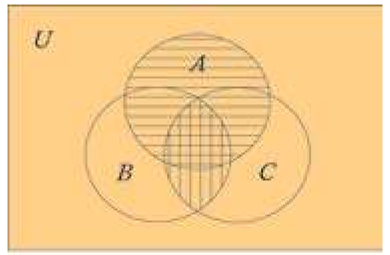


Fig (3)

$A \equiv$
$B \cap C$ |||
$A \cap (B \cap C$

In fig (4) doubly lined region represents
$(A \cap B) \cap C$.
Since in fig (3) and (4) these regions are the same therefore,
$A \cap (B \cap C) = (A \cap B) \cap C$.

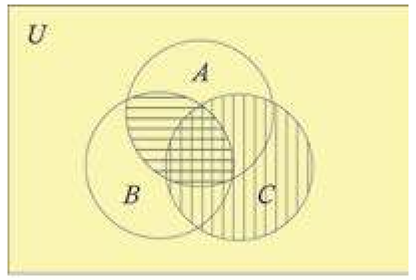


Fig (4)

$A \cap B \equiv$
C |||
$(A \cap B) \cap C$
#

(v) in fig. (5) $A \cup (B \cap C)$ is represented by the region which is lined horizontally or vertically or both ways.

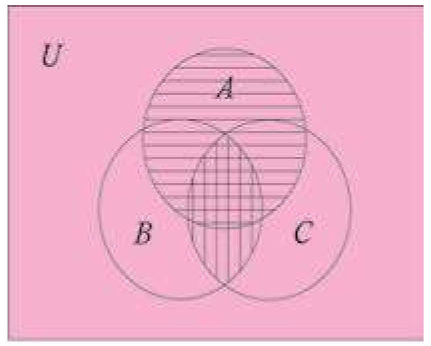


Fig (5)

$A \equiv$
$B \cap C$ |||
$A \cup (B \cap C)$
$\equiv$, ||| $\vee$ #

In fig. (6) $(A \cup B) \cap (A \cup C)$ is represented by the doubly lined region. Since the two region in fig (5) and (6) are the same, therefore

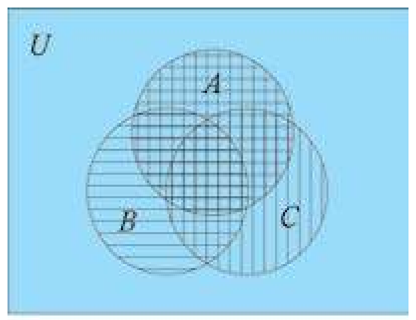


Fig (6)

$A \cup B \equiv$
$A \cup C$ |||
$(A \cup B) \cap (A \cup C)$
#

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(vi) Verify yourselves.

(vii) In fig (7) $(A \cup B)'$ is represented by vertically lined region.

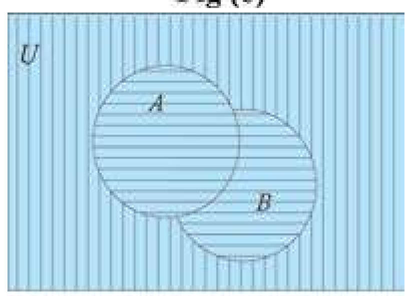


Fig (7)

$A \cup B \equiv$
$(A \cup B)'$ |||

In fig. (8) doubly lined region represents.
$A' \cap B'$.
The two regions in fig (7). And (8) are the same, therefore, $(A \cup B)' = A' \cap B'$

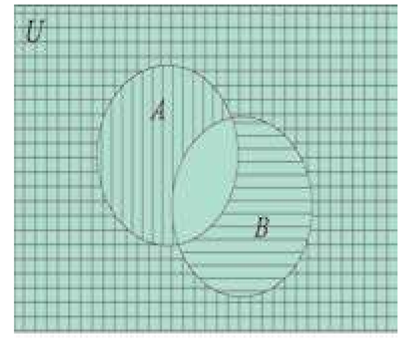


Fig (8)

$A' \equiv$
B'
$A' \cap B'$
#

(viii) Verify yourselves.

Note: In all the above Venn diagrams only overlapping sets have been considered. Verification in other cases can also be effected similarly. Detail of verification may be written by yourselves.

Exercise 2.3

1. Verify the commutative properties of union and intersection for the following pairs of sets: -

i) $A = \{1,2,3,4,5\}$, $B = \{4,6,8,10\}$ ii) N, Z

iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$, $B = \mathbb{R}$.

2. Verify the properties for the sets A, B and C given below: -

i) Associativity of Union ii) Associativity of intersection.

iii) Distributivity of Union over intersection.

iv) Distributivity of intersection over union.

a) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7,8\}$, $C = \{5,6,7,9,10\}$

b) $A = \Phi$, $B = \{0\}$, $C = \{0,1,2\}$

c) N, Z, Q

3. Verify De Morgan's Laws for the following sets:
$U = \{1,2,3, \ldots, 20\}$, $A = \{2,4,6, \ldots, 20\}$ and $B = \{1,3,5, \ldots, 19\}$.

4. Let U = The set of the English alphabet
$A = \{x \mid x$ is a vowel$\}$, $B = \{y \mid y$ is a consonant$\}$,
Verify De Morgan's Laws for these sets.

5. With the help of Venn diagrams, verify the two distributive properties in the following cases w.r.t union and intersection.

i) $A \subseteq B$, $A \cap C = \Phi$ and B and C are overlapping.

ii) A and B are overlapping, B and C are overlapping but A and C are disjoint.

6. Taking any set, say $A = \{1,2,3,4,5\}$ verify the following: -

i) $A \cup \Phi = A$ ii) $A \cup A = A$ iii) $A \cap A = A$

7. If $U = \{1,2,3,4,5,\ldots, 20\}$ and $A = \{1,3,5, \ldots, 19\}$, verify the following:-

i) $A \cup A' = U$ ii) $A \cap U = A$ iii) $A \cap A' = \Phi$

8. From suitable properties of union and intersection deduce the following results:

i) $A \cap (A \cup B) = A \cup (A \cap B)$ ii) $A \cup (A \cap B) = A \cap (A \cup B)$.

9. Using venn diagrams, verify the following results.

i) $A \cap B' = A$ iff $A \cap B = \Phi$ ii) $(A - B) \cup B = A \cup B$.

iii) $(A - B) \cap B = \Phi$ iv) $A \cup B = A \cup (A' \cap B)$.

2.6 Inductive and Deductive Logic

In daily life we often draw general conclusions from a limited number of observations or experiences. A person gets penicillin injection once or twice and experiences reaction soon afterwards. He generalises that he is allergic to penicillin. We generally form opinions about others on the basis of a few contacts only. This way of drawing conclusions is called **induction**.

Inductive reasoning is useful in natural sciences where we have to depend upon repeated experiments or observations. In fact greater part of our knowledge is based on induction.

On many occasions we have to adopt the opposite course. We have to draw conclusions from accepted or well-known facts. We often consult lawyers or doctors on the basis of their good reputation. This way of reasoning i.e., drawing conclusions from premises believed to be true, is called **deduction**. One usual example of deduction is: All men are mortal. We are men. Therefore, we are also mortal.

Deduction is much used in higher mathematics. In teaching elementary mathematics we generally resort to the inductive method. For instance the following sequences can be continued, inductively, to as many terms as we like:

i) 2,4,6,... ii) 1,4,9,... iii) 1,-1,2,-2,3,-3,...

iv) 1,4,7,... v) $\frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \ldots\ldots$ vi) $\frac{1}{10}, \frac{2}{100}, \frac{4}{1000}, \ldots\ldots.$

As already remarked, in higher mathematics we use the deductive method. To start with we accept a few statements (called postulates) as true without proof and draw as many conclusions from them as possible.

Basic principles of deductive logic were laid down by Greek philosopher, Aristotle. The illustrious mathematician Euclid used the deductive method while writing his 13 books of geometry, called Elements. Toward the end of the 17th century the eminent German mathematician, Leibniz, symbolized deduction. Due to this device deductive method became far more useful and easier to apply.

2.6.1 Aristotelian and non-Aristotelian logics

For reasoning we have to use *propositions*. A daclarative statement which may be true or false but not both is called a proposition. According to Aristotle there could be only two possibilities - a proposition could be either true or false and there could not be any third possibility. This is correct so far as mathematics and other exact sciences are concerned. For instance, the statement $a = b$ can be either true or false. Similarly, any physical or chemical theory can be either true or false. However, in statistical or social sciences it is sometimes not possible to divide all statements into two mutually exclusive classes. Some statements may be, for instance, undecided.

Deductive logic in which every statement is regarded as true or false and there is no other possibility, is called Aristotlian Logic. Logic in which there is scope for a third or fourth possibility is called non-Aristotelian. we shall be concerned at this stage with Aristotelian logic only.

2.6.2 Symbolic logic

For the sake of brevity propositions will be denoted by the letters p , q etc. We give a brief list of the other symbols which will be used.

Symbol	How to be read	Symbolic expression	How to be read
$\sim$	not	$\sim p$	Not p, negation of p
$\wedge$	and	$p \wedge q$	p and q
$\vee$	or	$p \vee q$	p or q
$\rightarrow$	If... then, implies	$p \rightarrow q$	If p then q p implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q p is equivalent to q

Explanation of the use of the Symbols:

1) Negation: If p is any proposition its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false and if p is false, $\sim p$ is true. The adjoining table, called truth table, gives the possible truth- values of p and $\sim p$.

p	$\sim p$
T	F
F	T

Table (1)

2) Conjunction of two statements p and q is denoted symbolically as $p \wedge q$ (p and q). A conjunction is considered to be true only if both its components are true. So the truth table of $p \wedge q$ is table (2).

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table (2)

Example 1:

i) Lahore is the capital of the Punjab and Quetta is the capital of Balochistan.

ii) $4 < 5 \wedge 8 < 10$

iii) $4 < 5 \wedge 8 > 10$

iv) $2 + 2 = 3 \wedge 6 + 6 = 10$

Clearly conjunctions (i) and (ii) are true whereas (iii) and (iv) are false.

3) Disjunction of p and q is p or q. It is symbolically written $p \vee q$. The disjunction $p \vee q$ is considered to be true when at least one of the components p and q is true. It is false when both of them are false. Table (3) is the truth table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table (3)

Example 2:

i) 10 is a positive integer or π is a rational number. Find truth value of this disjunction.

Solution: Since the first component is true, the disjunction is true.

ii) A triangle can have two right angles or Lahore is the capital of Sind.

Solution: Both the components being false, the composite proposition is false.

2.7 Implication or conditional

A compound statement of the form if p then q, also written ***p* implies *q***, is called a **conditional** or an **implication**, p is called the **antecedent** or **hypothesis** and q is called the **consequent** or the **conclusion**.

A conditional is regarded as false only when the antecedent is true and consequent is false. In all other cases it is considered to be true. Its truth table is, therefore, of the adjoining form.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table (4)

Entries in the first two rows are quite in consonance with common sense but the entries of the last two rows seem to be against common sense. According to the third row the conditional

If p then q

is true when p is false and q is true and the compound proposition is true (according to the fourth row of the table) even when both its components are false. We attempt to clear the position with the help of an example. Consider the conditional

If a person A lives at Lahore, then he lives in Pakistan.

If the antecedent is false i.e., A does not live in Lahore, all the same he may be living in Pakistan. We have no reason to say that he does not live in Pakistan.

We cannot, therefore, say that the conditional is false. So we must regard it as true. It must be remembered that we are discussing a problem of Aristotlian logic in which every proposition must be either true or false and there is no third possibility. In the case under discussion there being no reason to regard the proposition as false, it has to be regarded as true. Similarly, when both the antecedent and consequent of the conditional under consideration are false, there is no justification for quarrelling with the proposition. Consider another example.

A certain player, *Z*, claims that if he is appointed captain, the team will win the tournament. There are four possibilities: -

i) *Z* is appointed captain and the team wins the tournament. *Z*'s claim is true.

ii) *Z* is appointed captain but the team loses the tournament. *Z*'s claim is falsified.

iii) *Z* is not appointed captain but the team all the same wins the tournament.There is no reason to falsify *Z*'s claim.

iv) *Z* is not appointed captain and the team loses the tournament. Evidently, blame cannot be put on *Z*.

It is worth noticing that emphasis is on the conjunction if occurring in the beginning of the ancedent of the conditional. If condition stated in the antecedent is not satisfied we should regard the proposition as true without caring whether the consequent is true or false.

For another view of the matter we revert to the example about a Lahorite:

'If a person A lives at Lahore, then he lives in Pakistan'.

p: A person A lives at Lahore.

q: He lives in Pakistan

When we say that this proposition is true we mean that in this case it is not possible that '*A* lives at Lahore' is true and that '*A* does not live in Pakistan' is also true, that is $p \to q$ and $\sim(p \wedge \sim q)$ are both simultaneously true. Now the truth table of $\sim(p \wedge \sim q)$ is shown below:

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Table (5)

Looking at the last column of this table we find that truth values of the compound proposition $\sim(p \wedge \sim q)$ are the same as those adopted by us for the conditional $p \to q$. This shows that the two propositions $p \to q$ and $\sim(p \wedge \sim q)$ are logically equivalent. Therefore, the truth values adopted by us for the conditional are correct.

2.7.1 Biconditional : $p \leftrightarrow q$

The proposition $p \to q \wedge q \to p$ i s shortly written $p \leftrightarrow q$ and is called the **biconditional** or **equivalence**. It is read ***p*** iff ***q*** (iff stands for "if and only if ')

We draw up its truth table.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table (6)

From the table it appears that $p \leftrightarrow q$ is true only when both p and q are true or both p and q are false.

2.7.2 Conditionals related with a given conditional.

Let $p \to q$ be a given conditional. Then

i) $q \to p$ is called the **converse** of $p \to q$;

ii) $\sim p \to \sim q$ is called the **inverse** of $p \to q$;

iii) $\sim q \to \sim p$ is called the **contrapositive** of $p \to q$.

To compare the truth values of these new conditionals with those of $p \to q$ we draw up their joint table.

				Given conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	F	T	T	T	T

Table (7)

From the table it appears that

i) Any conditional and its contrapositive are equivalent therefore any theorem may be proved by proving its contrapositive.

ii) The converse and inverse are equivalent to each other.

Example 3: Prove that in any universe the empty set ϕ is a subset of any set A.

First Proof: Let U be the universal set consider the conditional:

$$\forall x \in U,\ x \in \phi \rightarrow x \in A \qquad (1)$$

The antecedent of this conditional is false because no $x \in U$, is a member of ϕ.
Hence the conditional is true.

Second proof: (By contrapositive)
The contrapositive of conditional (1) is

$$\forall x \in U,\ x \notin A \rightarrow x \notin \phi \qquad (2)$$

The consequent of this conditional is true. Therefore, the conditional is true.
Hence the result.

Example 4: Construct the truth table ot $[(p \rightarrow q) \wedge p \rightarrow q]$

Solution : Desired truth table is given below: -

p	p	$p \rightarrow q$	$(p \rightarrow q) \wedge\ p$	$[(p \rightarrow q) \wedge p \rightarrow q]$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table (8)

2.7.3 Tautologies

i) A statement which is true for all the possible values of the variables involved in it is

called a tautology, for example, $p \rightarrow q \leftrightarrow (\sim q \rightarrow \sim p)$ is a *tautology*.(are already verified by a truth table).

ii) A statement which is always false is called an **absurdity** or a contradiction e.g., $p \rightarrow \sim p$

iii) A statement which can be true or false depending upon the truth values of the variables involved in it is called a **contingency** e.g., $(p \rightarrow q) \wedge (p \vee q)$ is a contingency.
(You can verify it by constructing its truth table).

2.7.4 Quantifiers

The words or symbols which convey the idea of quantity or number are called quantifiers.
In mathematics two types of quantifiers are generally used.

i) **Universal quantifier** meaning for all
Symbol used : $\forall$

ii) **Existential quantifier:** There exist (some or few, at least one) symbol used: $\exists$

Example 5:

i) $\forall x \in A$, $p(x)$ is true.
(To be read : For all x belonging to A the statement $p(x)$ is true).

ii) $\exists x \in A \ni p(x)$ is true.
(To be read : There exists x belonging to A such that statement $p(x)$ is true).

The symbol $\ni$ stands for such that

Exercise 2.4

1. Write the converse, inverse and contrapositive of the following conditionals: -
   i) $\sim p \rightarrow q$ ii) $q \rightarrow p$ iii) $\sim p \rightarrow \sim q$ iv) $\sim q \rightarrow \sim p$

2. Construct truth tables for the following statements: -
   i) $(p \rightarrow \sim p) \vee (p \rightarrow q)$ ii) $(p \wedge \sim p) \rightarrow q$

iii) $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$

3. Show that each of the following statements is a tautology: -
   i) $(p \wedge q) \to p$ ii) $p \to (p \vee q)$
   iii) $\sim(p \to q) \to p$ iv) $\sim q \wedge (p \to q) \to \sim p$

4. Determine whether each of the following is a tautology, a contingency or an absurdity: -
   i) $p \wedge \sim p$ ii) $p \to (q \to p)$ iii) $q \vee (\sim q \vee p)$
5. Prove that $p \vee (\sim p \wedge \sim q) \vee (p \wedge q) = p \vee (\sim p \wedge \sim q)$

2.8 Truth Sets, A link between Set Theory and Logic.

Logical propositions p, q etc., are formulae expressed in terms of some variables. For the sake of simplicity and convenience we may assume that they are all expressed in terms of a single variable x where x is a real variable. Thus $p = p(x)$ where, $x \in \mathbb{R}$. All those values of x which make the formula $p(x)$ true form a set, say P. Then P is the truth set of p. Similarly, the **truth set**, Q, of q may be defined. We can extend this notion and apply it in other cases.

i) **Truth set of ~p:** Truth set of $\sim p$ will evidently consist of those values of the variable for which p is false i.e., they will be members of P', the complement of P.

ii) **$p \vee q$:** Truth set of $p \vee q = p(x) \vee q(x)$ consists of those values of the variable for which $p(x)$ is true or $q(x)$ is true or both $p(x)$ and $q(x)$ are true.

Therefore, truth set of $p \vee q$ will be:

$P \cup Q \quad = \{x \mid p(x)$ is true or $q(x)$ is true$\}$

iii) **$p \wedge q$:** Truth set of $p(x) \wedge q(x)$ will consist of those values of the variable for which both $p(x)$ and $q(x)$ are true. Evidently truth set of

$p \wedge q = P \cap Q$

$= \{x \mid p(x)$ is true $\wedge\, q(x)$ is true$\}$

iv) **$p \to q$:** We know that $p \to q$ is equivalent to $\sim p \vee q$ therefore truth set of $p \to q$ will be $P' \cup Q$

v) **$p \leftrightarrow q$:** We know that $p \leftrightarrow q$ means that p and q are simultaneously true or false. Therefore, in this case truth sets of p and q will be the same i.e.,

$$P = Q$$

Note: **(1)** Evidently truth set of a tautology is the relevant universal set and that of an absurdity is the empty set Φ.

(2) With the help of the above results we can express any logical formula in set-theoretic form and vice versa.

We will illustrate this fact with the help of a solved example.

Example 1: Give logical proofs of the following theorems: -

(A, B and C are any sets)

i) $(A \cup B)' = A' \cap B'$ ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution: i) The corresponding formula of logic is

$\sim(p \vee q) = \sim p \wedge \sim q$ (1)

We construct truth table of the two sides.

p	p	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

The last two columns of the table establish the equality of the two sides of eq.(1)

(ii) Logical form of the theorem is

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

We construct the table for the two sides of this equation

1	2	3	4	5	6	7	8
p	p	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Comparison of the entries of columns 5 and 8 is sufficient to establish the desired result.

Exercise 2.5

Convert the following theorems to logical form and prove them by constructing truth tables: -

1. $(A \cap B)' = A' \cup B'$ **2.** $(A \cup B) \cup C = A \cup (B \cup C)$

3. $(A \cap B) \cap C = A \cap (B \cap C)$ **4.** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2.9 Relations

In every-day use *relation* means an abstract type of connection between two persons or objects , for instance, (Teacher, Pupil), (Mother, Son), (Husband, Wife), (Brother, Sister), (Friend, Friend), (House, Owner). In mathematics also some operations determine relationship between two numbers, for example: -

$>$: (5 , 4); square: (25, 5); Square root: (2,4); Equal: (2×2, 4).

Technically a *relation* is a set of ordered pairs whose elements are ordered pairs of related numbers or objects. The relationship between the components of an ordered pair may or may not be mentioned.

i) Let A and B be two non-empty sets, then any subset of the Cartesian product $A \times B$ is called a **binary relation**, or simply a **relation**, from A to B. Ordinarily a relation will be denoted by the letter r.

ii) The set of the first elements of the ordered pairs forming a relation is called its **domain**.

iii) The set of the second elements of the ordered pairs forming a relation is called its **range**.

iv) If A is a non-empty set, any subset of $A \times A$ is called a **relation in *A***. Some authors call it a **relation on *A***.

Example 1: Let c_1, c_2, c_3 be three children and m_1, m_2 be two men such that father of both c_1, c_2 is m_1 and father of c_3 is m_2. Find the relation {(child, father)}

Solution: C = Set of children = $\{c_1, c_2, c_3\}$ and F = set of fathers = $\{m_1, m_2\}$

$C \times F = \{(c_1, m_1), (c_1, m_2), (c_2, m_1), (c_2, m_2), (c_3, m_1), (c_3, m_2)\}$

r = set of ordered pairs (*child, father*).

$= \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

Dom $r = (c_1, c_2, c_3\}$, Ran $r = \{m_1, m_2\}$

The relation is shown diagrammatically in fig. (2.29).

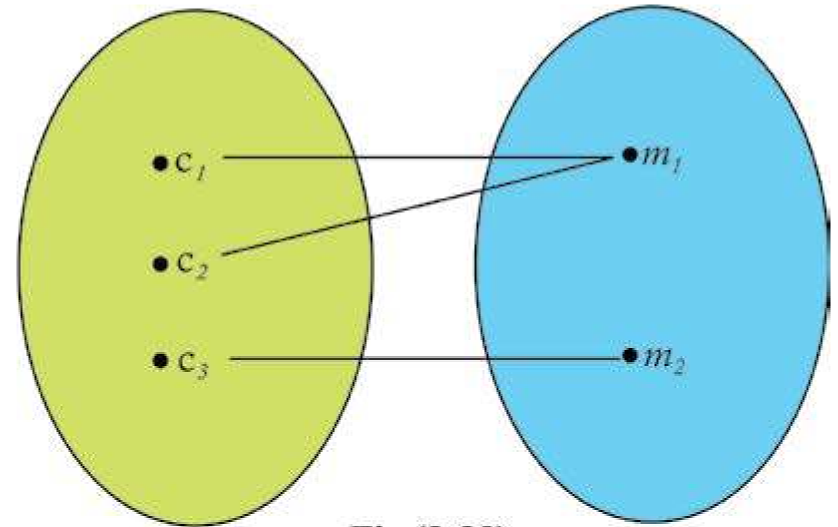


Fig (2.29)

Example 2: Let $A = \{1, 2, 3\}$. Determine the relation r such that xry iff $x < y$.

Solution: $A \times A = \{(1, 1),(1, 2),(1, 3),(2, 1), (2, 2),(2, 3),(3, 1),(3, 2),(3, 3)\}$

Clearly, required relation is:

$r = \{(1, 2), (1, 3), (2, 3)\}$, *Dom* $r = \{1, 2\}$, Ran $r = \{2, 3\}$

Example 3: Let $A = \mathbb{R}$, the set of all real numbers.

Determine the relation r such that $x\ r\ y$ iff $y = x + 1$

Solution: $A \times A = \mathbb{R} \times \mathbb{R}$

$r = \{ (x,y) | \ y = x+1\}$

When $x = 0$, $y = 1$

$x = -1$, $y = 0$,

r is represented by the line passing through the points (0,1), (– 1,0).

Some more points belonging to r are:
$\{(1, 2), (2, 3), (3, 4), (-2, -1), (-3, -2), (-4, -3)\}$
Clearly, Dom $r = \mathbb{R}$, and Ran $r = \mathbb{R}$

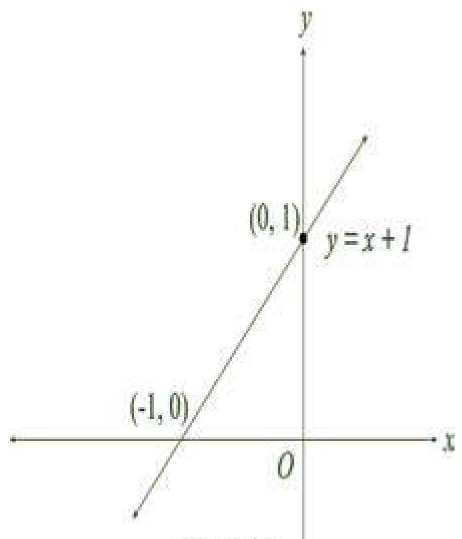


Fig (2.30)

2.10 Functions

A very important special type of relation is a function defined as below: -
Let A and B be two non-empty sets such that:

i) f is a relation from A to B that is, f is a subset of $A \times B$
ii) Dom $f = A$
iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function f is also written as:

$$f : A \to B$$

which is read: f is a function from A to B.
If (x, y) in an element of f when regarded as a set of ordered pairs, we write $y = f(x)$. y is called the value of f for x or image of x under f.
In example 1 discussed above

i) r is a subset of $C \times F$
ii) Dom $r = \{c_1, c_2, c_3\} = C$;
iii) First elements of no two related pairs of r are the same.

Therefore, r is a function from C to F.
In Example 2 discussed above

i) r is a subset of $A \times A$;
ii) Dom $r \neq A$

Therefore, the relation in this case is not a function.
In example 3 discussed above

i) r is a subset of $\mathbb{R}$

ii) Dom $r = \mathbb{R}$
iii) Clearly first elements of no two ordered pairs of r can be equal. Therefore, in this case r is a function.

i) **Into Function:** If a function $f : A \to B$ is such that Ran $f \subset B$ i.e., Ran $f \neq B$, then f is said to be a function from A **into** B. In fig.(1) f is clearly a function. But Ran $f \neq B$. Therefore, f is a function from A into B.

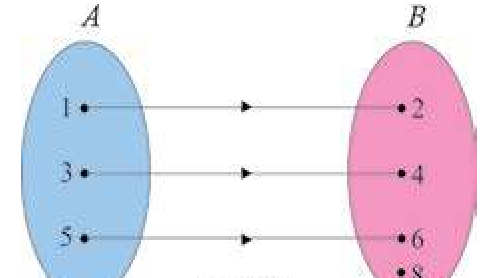


Fig (1)
$f = \{(1,2), (3,4), (5,6)\}$

ii) **Onto (Surjective) function:** If a function $f : A \to B$ is such that Ran $f = B$ i.e., every element of B is the image of some elements of A, then f is called an **onto** function or a surjective function.

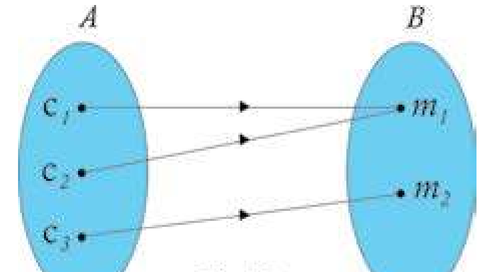


Fig (2)
$f = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

iii) **(1-1) and into (Injective) function:** If a function f from A into B is such that second elements of no two of its ordered pairs are equal, then it is called an injective (1 - 1, and into) function. The function shown in fig (3) is such a function.

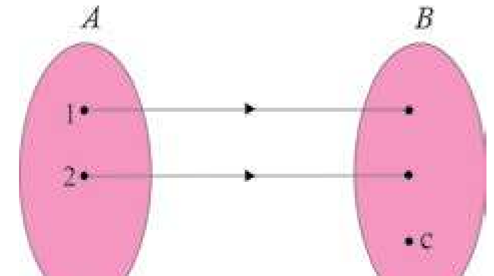


Fig (3)
$f = \{(1, a)(2, b)\}$

iv) **(1 –1) and Onto function (bijective function).** If f is a function from A onto B such that second elements of no two of its ordered pairs are the same, then f is said to be (1 – 1) function from A onto B.
Such a function is also called a (1 – 1) correspondence between A and B. It is also called a bijective function.
Fig(4) shows a (1-1) correspondence between the sets A and B.

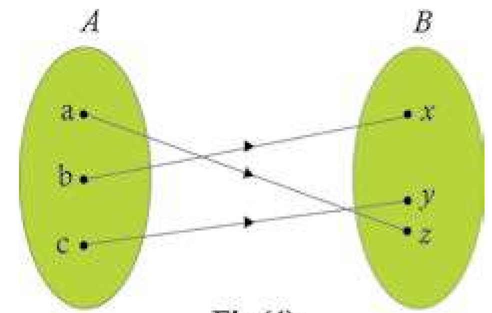


Fig (4)
$f = \{(a, z), (b, x), (c, y)\}$

(a, z), (b, x) and (c, y) are the pairs of corresponding elements i.e., in this case $f = \{(a, z), (b, x), (c, y)\}$ which is a bijective function or (1 - 1) correspondence between the sets A and B.

Set - Builder Notation for a function: We know that set-builder notation is more suitable for infinite sets. So is the case in respect of a function comprising infinite number of ordered pairs. Consider for instance, the function

$f = \{ (1,1), (2,4), (3, 9), (4, 16),...\}$

Dom $f = (1, 2, 3,4, ...\}$.and Ran $f = \{1,4,9, 16, ...\}$

This function may be written as: $f = \{(x, y) \mid y = x^2, x \in N\}$

For the sake of brevity this function may be written as:

f = function defined by the equation $y = x^2$, $x \in N$

Or, to be still more brief: The function x^2, $x \in N$

In algebra and Calculus the domain of most functions is $\mathbb{R}$ and if evident from the context it is, generally, omitted.

2.10.1 Linear and Quadratic Functions

The function $\{(x, y) \mid y = mx + c\}$ is called a **linear function**, because its graph (geometric representation) is a straight line. Detailed study of a straight line will be undertaken in the next class. For the present it is sufficient to know that an equationof the form

$y = mx + c$ or $ax + by + c = 0$ represents a straight line . This can be easily verified by drawing graphs of a few linear equations with numerical coefficients. The function $\{(x, y) | y = ax^2 + bx + c\}$ is called a **quadratic function** because it is defined by a quadratic (second degree) equation in x, y.

Example 4: Give rough sketch of the functions

i) $\{(x, y) \mid 3x + y = 2\}$ ii) $\{(x, y) \mid y = \frac{1}{2}x^2\}$

Solution:

i) The equation defining the function is $3x + y = 2$

$\Rightarrow y = -3x + 2$

We know that this equation, being linear, represents a straight line. Therefore, for drawing its sketch or graph only two of its points are sufficient.

When $x = 0, y = 2$,

When $y = 0, x = \frac{2}{3} = 0.6$ nearly. So two points on the line are A (0, 2) and B = (0.6, 0).

Joining A and B and producing $\overline{AB}$ in both directions, we obtain the line AB i.e., graph of the given function.

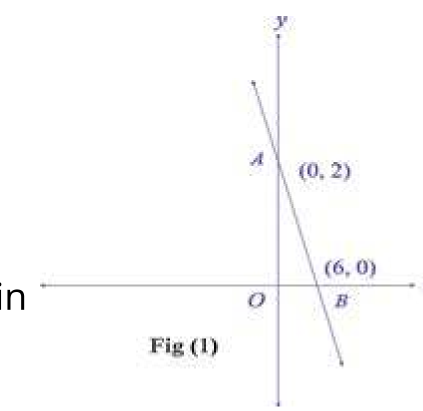


Fig (1)

ii) The equation defining the function is $y = \frac{1}{2}x^2$.

Corresponding to the values 0, ± 1, ±2, ±3 ... of x, values of y are 0, .5, 2, 4.5, ...

We plot the points (0, 0), (±1, .5), (±2, 2), (±3, 4.5), ...

Joining them by means of a smooth curve and extending it upwards we get the required graph. We notice that:

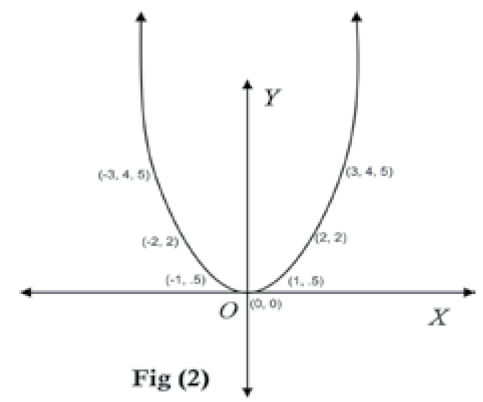


Fig (2)

i) The entire graph lies above the x-axis.

ii) Two equal and opposite values of x correspond to every value of y (but not vice versa).

iii) As x increases (numerically) y increases and there is no end to their ncrease. Thus the graph goes infinitely upwards. Such a curve is called a parabola. The students will learn more about it in the next class.

2.11 Inverse of a function

If a relation or a function is given in the tabular form i.e., as a set of ordered pairs, its inverse is obtained by interchanging the components of each ordered pair. The inverse of r and f are denoted r^{-1} and f^{-1} respectively.

If r or f are given in set-builder notation the inverse of each is obtained by interchanging x and y in the defining equation. The inverse of a function may or may not be a function.

The inverse of the linear function

$\{(x, y) \mid y = mx + c\}$ is $\{(x, y) \mid x = my + c\}$ which is also a linear function. Briefly, we may say that the **inverse of a line is a line.**

The line $y = x$ is clearly self-inverse. The function defined by this equation i.e., the function $\{(x, y) \mid y = x\}$ is called the identity function.

Example 6: Find the inverse of

i) $\{(1, 1), (2,4), (3,9), (4, 16), \ldots x \in Z^+\}$,

ii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$ iii) $\{(x, y) \mid x^2 + y^2 = a^2\}$.

Tell which of these are functions.

Solution:

i) The inverse is:

$\{(2,1), (4, 2), (9, 3), (16, 4) \ldots\}$.

This is also a function.

Note: Remember that the equation

$y = \sqrt{x}, \; x \geq 0$

defines a function but the equation $y^2 = x, x \geq 0$ does not define a function.

The function defined by the equation

$y = \sqrt{x}, \; x \geq 0$

is called the **square root function.**

The equation $y^2 = x \Rightarrow y = \pm\sqrt{x}$

Therefore, the equation $y^2 = x \; (x \geq 0)$ may be regarded as defining the union of the functions defined by

$y = \sqrt{x}, \; x \geq 0$ and $y = -\sqrt{x}, \; x \geq 0$.

ii) The given function is a **linear function**. Its inverse is:

$\{(x, y) \mid x = 2y + 3\}$

which is also a linear function.

Points $(0, 3)$, $(-1.5, 0)$ lie on the given line and points $(3, 0)$, $(0, -1.5)$ lie on its inverse. (Draw the graphs yourselves).

The lines l, l' are symmetric with respect to the line $y = x$. This quality of symmetry is true not only about a linear n function and its inverse but is also true about any function of a higher degree and its inverse (why?).

Exercise 2.6

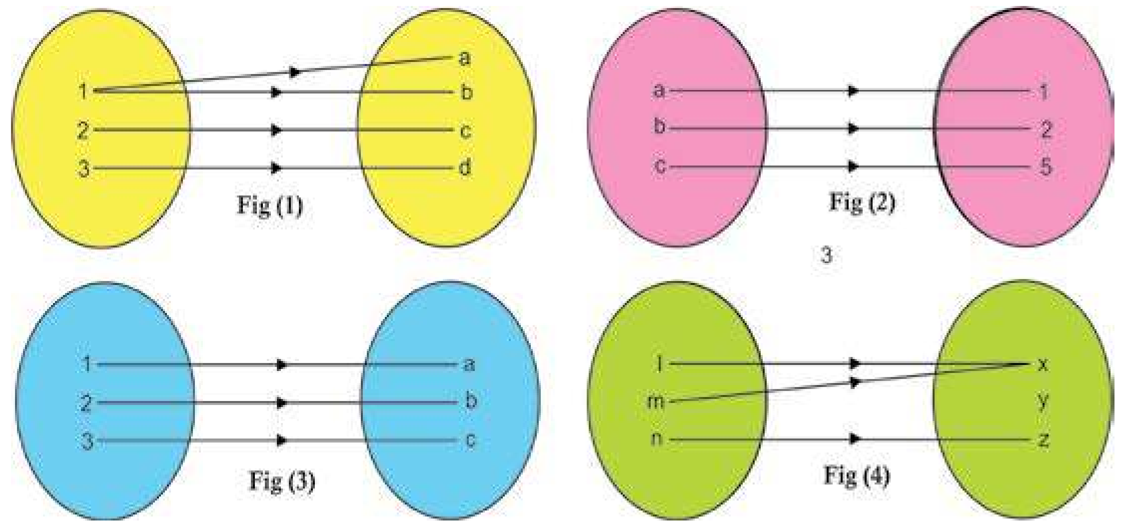
1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation. Also draw the graph of each.

i) $\{(x, y) \mid y = x\}$ ii) $\{(x, y) \mid y + x = 5\}$

ii) $\{(x, y) \mid x + y < 5\}$ iv) $\{(x, y) \mid x + y > 5\}$

2. Repeat Q -1 when $A = \mathbb{R}$, the set of real numbers. Which of the real lines are functions.

3. Which of the following diagrams represent functions and of which type?



4. Find the inverse of each of the following relations. Tell whether each relation and its inverse is a function or not: -

i) $\{(2,1), (3,2), (4,3), (5,4), (6,5)\}$ ii) $\{(1,3), (2,5), (3,7), (4,9), (5,11)\}$

iii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$ iv) $\{(x, y) \mid y^2 = 4ax, x \geq 0\}$

v) $\{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

2.12 Binary Operations

In lower classes we have been studying different number systems investigating the properties of the operations performed on each system. Now we adopt the opposite course. We now study certain operations which may be useful in various particular cases.

An operation which when performed on a single number yields another number of the same or a different system is called a **unary operation.**

Examples of *Unary operations* are negation of a given number, extraction of square roots or cube roots of a number, squaring a number or raising it to a higher power.

We now consider binary operation, of much greater importance, operation which requires two numbers. We start by giving a formal definition of such an operation.

A *binary operation* denoted as ※ (read as star) on a non-empty set G is a function which associates with each ordered pair (a, b), of elements of G, a unique element, denoted as a ※ b of G.

In other words, a binary operation on a set G is a function from the set $G \times G$ to the set G. For convenience we often omit the word *binary* before *operation*.

Also in place of saying ※ is an operation on G, we shall say G is closed with respect to ※ .

Example 1: Ordinary addition, multiplication are operations on N. i.e., N is closed with respect to ordinary addition and multiplication because

$$\forall\, a, b \in N,\ a + b \in N \wedge a.b \in N$$

($\forall$ stands for" for all" and $\wedge$ stands for" and")

Example 2: Ordinary addition and multiplication are operations on E, the set of all even natural numbers. It is worth noting that addition is not an operation on O, the set of old natural numbers.

Example 3: With obvious modification of the meanings of the symbols, let E be any even natural number and O be any odd natural number, then

$E \oplus E = E$ (Sum of two even numbers is an even number).

$E \oplus O = O$

and $O \oplus O = E$

$\oplus$	E	O
E	E	O
O	O	E

These results can be beautifully shown in the form of a table given above:

This shows that the set $\{E, O\}$ is closed under (ordinary) addition.

The table may be read (horizontally).

$E \oplus E = E,\qquad E \oplus O = O;$

$O \oplus O = E,\qquad O \oplus E = O$

Example 4: The set $\{1, -1, i, -i\}$ where $i = \sqrt{-1}$ is closed w.r.t multiplication (but not *w. r. t* addition). This can be verified from the adjoining table.

$\otimes$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note: The elements of the set of this example are the **fourth roots** of unity.

Example 5: It can be easily verified that ordinary multiplication (but not addition) is an operation on the set $\{1, \omega, \omega^2\}$ where $\omega^3 = 1$. The adjoining table may be used for the verification of this fact.

(ω is pronounced omega)

$\otimes$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Operations on Residue Classes Modulo *n*.

Three consecutive natural numbers may be written in the form:

$3n$, $3n + 1$, $3n + 2$ When divided by 3 they give remainders 0, 1, 2 respectively.

Any other number, when divided by 3, will leave one of the above numbers as the reminder. On account of their special importance (in theory of numbers) the remainders like the above are called **residue classes Modulo 3.** Similarly, we can define **Residue classes Modulo 5** etc. An interesting fact about residue classes is that ordinary addition and multiplication are operations on such a class.

Example 6: Give the table for addition of elements of the set of residue classes modulo 5.

Solution: Clearly $\{0,1,2,3,4\}$ is the set of residues that we have to consider. We add pairs of elements as in ordinary addition except that when the sum equals or exceeds 5, we divide it out by 5 and insert the remainder only in the table. Thus $4 + 3 = 7$ but in place of 7 we insert $2(= 7-5)$ in the table and in place of $2 + 3 = 5$, we insert $0(= 5-5)$.

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Example 7: Give the table for addition of elements of the set of residue classes modulo 4.

Solution: Clearly $\{0,1,2,3\}$is the set of residues that we have to consider. We add pairs of elements as in ordinary addition except that when the sum equals or exceeds 4, we divide it out by 4 and insert the remainder only in the table. Thus $3 + 2 = 5$ but in place

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

of 5 we insert 1(= 5−4) in the table and in place of 1 + 3 = 4 , we insert 0(= 4−4).

Example 8: Give the table for multiplication of elemnts of the set of residue classes modulo 4.

Solution: Clearly {0,1,2,3} is the set of residues that we have to consider. We multiply pairs of elements as in ordinary multiplcation except that when the product equals or exceeds 4, we divide it out by 4 and insert the remainder only in the table. Thus 3×2=6 but in place of 6 we insert 2 (= 6−4)in the table and in place of 2×2=4, we insert 0(= 4−4).

⊗	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Example 9: Give the table for multiplication of elements of the set of residue classes modulo 8.

Solution: Table is given below:

⊗	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Note: For performing multiplication of residue classes 0 is generally omitted.

2.12.1 Properties of Binary Operations

Let S be a non-empty set and ※ a binary operation on it. Then ※ may possess one or more of the following properties: -

i) Commutativity: ※ is said to be commutative if

$$a ※ b = b ※ a \;\forall\; a,b \in S.$$

ii) Associativity: ※ is said to be associative on S if

$$a ※ (b ※ c) = (a ※ b) ※ c \;\forall\; a,b,c \in S.$$

iii) Existence of an identity element: An element $e \in S$ is called an identity element w.r.t ※ if

$$a ※ e = e ※ a = a, \;\forall\; a \in S.$$

iv) Existence of inverse of each element: For any element $a \in S, \exists$ an element $a' \in S$ such that

$$a ※ a' = a' ※ a = e \qquad \text{(the identity element)}$$

Note: **(1)** The Symbol ∃ stands for 'there exists'.

(2) Some authors include closure property in the properties of an operation. Since this property S is already included in the definition of *operation* we have considered it unnecessary to mention it in the above list.

(3) Some authors define *left identity* and *right identity* and also *left inverse* and *right inverse* of each element of a set and prove uniqueness of each of them. The following theorem gives their point of view: -

Theorem:

i) In a set S having a binary operation ※ a left identity and a right identity are the same.

ii) In a set having an associative binary Operation left inverse of an element is equal to its right inverse.

Proof:

ii) Let e' be the left identity and e'' be the right identity. Then

$$e' ※ e'' = e' \qquad (\because e'' \text{ is a right identity})$$
$$= e'' \qquad (\because e' \text{ is a left identity})$$

Hence $e' = e'' = e$

Therefore, e is the unique identity of S under ※

ii) For any $a \in S$, let a', a'' be its left and right inverses respectively then

$$a' ※ (a ※ a'') = a' ※ e \qquad (\because a'' \text{ is right inverse of } a)$$
$$= a' \qquad (\because e \text{ is the identity})$$

Also $$(a' ※ a) ※ a'' = e ※ a'' \qquad (\because a' \text{ is left inverse of } a)$$
$$= a''$$

But $a' ※ (a ※ a'') = (a' ※ a) ※ a''$ ※ is associative as supposed)

$$\therefore \; a' = a''$$

Inverse of a is generally written as a^{-1}.

Example 10: Let A =(1,2,3,...., 20}, the set of first 20 natural numbers.
Ordinary addition is not a binary operation on A because the set is not closed w.r.t. addition. For instance, $10 + 25 = 25 \notin A$

Example 11: Addition and multiplication are commutative and associative operations on the sets

$N, Z, Q, \mathbb{R}$, (usual notation),

e.g. $4 \times 5 = 5 \times 4$, $2 + (3 + 5) = (2 + 3) + 5$ etc.

Example 12: Verify by a few examples that subtraction is not a binary operation on N but it is an operation on Z, the set of integers.

Exercise 2.7

1. Complete the table, indicating by a tick mark those properties which are satisfied by the specified set of numbers.

Set of numbers → / Property ↓		Natural	Whole	Integers	Rational	Reals
Closure	$\oplus$					
	$\otimes$					
Associative	$\oplus$					
	$\otimes$					
Identity	$\oplus$					
	$\otimes$					
Inverse	$\oplus$					
	$\otimes$					
Commutative	$\oplus$					
	$\otimes$					

2. What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers?

3. Show that the adjoining table is that of multiplication of the elements of the set of residue classes modulo 5.

※	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

4. Prepare a table of addition of the elements of the set of residue classes modulo 4.
5. Which of the following binary operations shown in tables (a) and (b) is commutative?

※	a	b	c	d
a	a	c	b	d
b	b	c	b	a
c	c	d	b	c
d	a	a	b	b

(a)

※	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

(b)

6. Supply the missing elements of the third row of the given table so that the operation ※ may be associative.

※	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	-	-	-	-
d	d	c	c	d

7. What operation is represented by the adjoining table? Name the identity element of the relevant set, if it exists. Is the operation associative? Find the inverses of 0,1,2,3, if they exist.

※	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2.13 Groups

We have considered, at some length, binary operations and their properties. We now use our knowledge to classify sets according to the properties of operations defined on them.

First we state a few preliminary definitions which will culminate in the definition of a group.

Groupoid: A **groupoid** is a non-empty set on which a binary operation $※$ is defined.

Some authors call the system $(S, ※)$ a groupoid. But, for the sake of brevity and convenience we shall call S a **groupoid**, it being understood that an operation $※$ is defined on it.

In other words, a closed set with respect to an operation $※$ is called a groupoid.

Example 1: The set $\{E, O\}$ where E is any even number and O is any odd number, (as already seen) are closed w.r.t. addition.

It is, therefore, a groupoid.

Example 2: The set of Natural numbers is not closed under operation of subtraction e.g.,

For $4, 5 \in N$, $4 - 5 = -1 \notin N$

Thus $(N, -)$ is not a groupoid under subtraction.

Example 3: As seen earlier with the help of a table the set $\{1, -1, i, -i\}$, is closed w.r.t. . multiplication (but not w.r.t. addition). So it is also a groupoid w.r.t $\times$.

Semi-group: A non-empty set S is semi-group if;

i) It is **closed** with respect to an operation $※$ and

ii) The operation $※$ is **associative.**

As is obvious from its very name, a semi-group satisfies half of the conditions required for a group.

Example 4: The set of natural numbers, N, together with the operation of addition is a semi-group. N is clearly closed w.r.t. addition (+).

Also $\forall\ a,b,c \in N$, $a + (b + c) = (a + b) + c$

Therefore, both the conditions for a semi-group are satisfied.

Non-commutative or non-abelian set: A set A is non-commutative if commutative law does not hold for it.

For example a set A is non-commutative or non-abilian set under $※$ when is defined as: $\forall\ x, y \in x ※ y = x$.

Clearly $x ※ y = x$ and $y ※ x = y$ indicates that A is non-commutative or non-abilian set.

Example 5: Consider Z, the set of integers together with the operation of multiplication. Product of any two integers is an integer.

Also product of integers is associative because $\forall a,b,c \in Z$ $\quad a.(b.c) = (a.b).c$

Therefore, $(Z, .)$ is a semi-group.

Example 6: Let $P(S)$ be the power-set of S and let $A, B, C,$ ... be the members of P. Since union of any two subsets of S is a subset of S, therefore P is closed with respect to $\cup$. Also the operation is associative.

(e.g. $A \cup (B \cup C) = (A \cup B) \cup C$, which is true in general),

Therefore, $(P(S), \cup)$ is a semi-group.

Similarly $(P(S), \cap)$ is a semi-group.

Example 7: Subtraction is non-commutative and non-associative on N.

Solution: For $4, 5, 6, \in N$, we see that

$4-5 = -1$ and $5-4 = 1$

Clearly $\quad 4-5 \neq 5-4$

Thus subtraction is non-commutative on N.

Also $5-(4-1) = 5-(3) = 2$ and $(5-4)-1 = 1-1 = 0$

Clearly $5-(4-1) \neq (5-4)-1$

Thus subtraction is non-associative on N.

Example 8: For a set A of distinct elements, the binary operation $※$ on A defined by

$x ※ y = x, \forall\ x, y \in A$

is non commutative and assocaitve.

Solution : Consider

$x ※ y = x$ $\quad$ and $y ※ x = y$

Clearly $\quad x ※ y \neq y ※ x$

Thus $※$ is non-commutative on A.

Monoid: A semi-group having an identity is called a **monoid** i.e., a monoid is a set S;

i) which is closed w.r.t. some operation $※$.

ii) the operation $※$ is associative and

iii) it has an identity.

Example 9: The power-set $P(S)$ of a set S is a **monoid** w.r.t. the operation $\cup$,because, as seen above, it is a semi-group and its identity is the empty-set Φ because if A is any subset of S,

$\Phi \cup A = A \cup \quad = A$

Example 10: The set of all non negative integers i.e., $Z^+ \cup \{0\}$

i) is clearly closed w.r.t. addition,
ii) addition is also associative, and
iii) 0 is the identity of the set.

$(a + 0 = 0 + a = a \quad \forall\ a \in Z^+ \cup \{0\})$

$\therefore$ the given set is a monoid w.r.t. addition.

Note: It is easy to verify that the given set is a monoid w.r.t. multiplication as well but not w.r.t. subtraction

Example 11: The set of natural numbers, N. w.r.t. $\otimes$

i) the product of any two natural numbers is a natural number;
ii) Product of natural numbers is also associative i.e.,

$\forall\ a,\ b,\ c \in N \quad a.(b.c) = (a.b).c$

iii) $1 \in N$ is the identity of the set.

$\therefore N$ is a monoid w.r.t. multiplication

Note: N is not a monoid w.r.t. addition because it has no identity w.r.t. addition.

Definition of Group: A *monoid* having inverse of each of its elements under ※ is called a group under ※. That is a group under ※ is a set G (say) if

i) G is closed w.r.t. some operation ※
ii) The operation of ※ is associative;
iii) G has an identity element w.r.t. ※ and
iv) Every element of G has an inverse in G w.r.t. ※.

If G satisfies the additional condition:

v) For every $a, b \in G$

a ※ $b = b$ ※ a

then G is said to be an Abelian* or commutative group under ※

Example 12: The set N w.r.t. +

Condition (i) colsure: satisfied i.e., $\forall\ a, b \in N, a + b \in N$

(ii) Associativity: satisfied i.e.,

$\forall\ a,b,c \in N,\ a + (b + c) = (a + b) + c$

(iii) and (iv) not satisfied i.e., neither identity nor inverse of any element exists.

$\therefore N$ is only a semi-group. Neither *monoid* nor a *group w.r.t.* +.

Example 13: N w.r.t $\otimes$

Condition: (i) Closure: satisfied

$\forall\ a, b \in N, \qquad a, b \in N$

(ii) Associativity: satisfied

$\forall\ a,b,c \in N, \quad a.(b.c) = (a.b).c$

(iii) Identity element, yes, 1 is the identity element

(iv) Inverse of any element of N does not exist in N, so N is a *monoid* but not a group under multiplication.

Example 14: Consider $S = \{0,1,2\}$ upon which operation $\oplus$has been performed as shown in the following table. Show that S is an abelian group under $\oplus$.

Solution :

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

i) Clearly S as shown under the operation is closed.
ii) The operation is associative e.g

$0 + (1 + 2) = 0 + 0 = 0$

$(0 + 1) + 2 = 1 + 2 = 0$ etc.

iii) Identity element 0 exists.
iv) Inverses of all elements exist, for example

$0 + 0 = 0,\ 1 + 2 = 0,\ 2 + 1 = 0$

$\Rightarrow 0^{-1} = 0 \quad 1^{-1} = 2, \ 2^{-1} = 1$

v) Also $\oplus$ is clearly commutative e.g., $1 + 2 = 0 = 2 + 1$

Hence the result,

Example 15: Consider the set $S = \{1, -1, i\ -i\}$. Set up its multiplication table and show that the set is an abelian group under multiplication

Solution :

$\otimes$	1	−1	i	$-i$
1	1	-1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

i) *S* is evidently closed w.r.t.$\otimes$.

ii) Multiplication is also associative

(Recall that multiplication of complex numbers is associative)

iii) Identity element of *S* is 1.

iv) Inverse of each element exists.

Each of 1 and -1 is self inverse.

i and - i are inverse of each other.

v) $\otimes$ is also commutative as in the case of *C*, the set of complex numbers. Hence given set is an *Abelian group*.

Example : Let *G* be the set of all 2×2 non-singular real matrices, then under the usual multiplication of matrices, *G* is a non-abelian group.

Condition (i) Closure: satisfied; i.e., product of any two 2×2 matrices is again a matrix of order 2×2.

(ii) Associativity: satisfied

For any matrices *A* , *B* and *C* conformable for multiplication.

$$A \times (B \times C) = (A \times B) \times C.$$

So, condition of associativity is satisfied for 2×2 matrices

(iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix.

(iv) As *G* contains non-singular matrices only so, it contains inverse of each of its elements.

(v) We know that $AB \neq BA$ in general. Particularly for G, $AB \neq BA$.

Thus *G* is a non-abilian or non-commutative gorup.

Finite and Infinite Gorup: A gorup G is said to be a **finite group** if it contains finite number of elements. Otherwise G is an **infinite group**.

The given examples of groups are clearly distinguishable whether finite or infinite.

Cancellation laws: If *a,b,c* are elements of a group *G*, then

i) $ab = ac \Rightarrow b = c$ (Left cancellation Law)

ii) $ba = ca \Rightarrow b = c$ (Right cancellation Law)

Proof: (i) $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ (by associative law)

$\Rightarrow eb = ec$ $(\because a^{-1}a = e)$

$\Rightarrow b = c$

ii) Prove yourselves.

2.14 Solution of linear equations

a,b being elements of a group *G*, solve the following equations:

i) $ax = b$, ii) $xa = b$

Solution : (i) Given: $ax = b \Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)x = a^{-1}b$ (by associativity)

$\Rightarrow ex = a^{-1}b$

$\Rightarrow x = a^{-1}b$ which is the desired solution.

ii) Solve yourselves.

Note: Since the inverse (left or right) of any element *a* of a group is unique, from the above procedure, it follows that the above solution is also unique.

2.15 Reversal law of inverses

If *a,b* are elements of a group *G*, then show that

$$(ab)^{-1} = b^{-1}a^{-1}$$

Proof: $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ (Associative law)

$= a\,e\,a^{-1}$

$= aa^{-1}$

$= e$

$\therefore a\,b$ and $b^{-1}a^{-1}$ are inverse of each other.

Note: The rule can obviously be extended to the product of three or more elements of a group.

Theorem: If (G, ※) is a group with e its identity, then e is unique.

Proof: Suppose the contrary that identity is not unique. And let e' be another identity.

e, e' being identities, we have

e' ※ e = e ※ e' = e' (e is an identity) (i)

e' ※ e = e ※ e' = e (e' i s an identity) (ii)

Comparing (i) and (ii)

$e' = e$.

Thus the identity of a group is always unique.

Examples:

i) $(Z, +)$ has no identity other then 0 (zero).

ii) $(\mathbb{R} - \{0\}, \times)$ has no identity other than 1.

iii) $(C,+)$ has no identity other than $0 + 0i$.

iv) $(C,.)$ has no identity other than $1 + 0i$.

v) $(M_2,.)$ has no identity other than $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

where M_2 is a set of 2×2 matrices.

Theoram: If (G, ※) is a group and $a \in G$, there is a unique inverse of a in G.

Proof: Let (G, ※) be a group and $a \in G$.

Suppose that a' and a'' are two inverses of a in G. Then

$a' = a'$ ※ $e = a'$ ※ (a ※ a'') ($a''$is an inverse of a w.r.t. ※)

= (a' ※ a) ※ a'' (Associative law in G).

= e ※ a'' (a' is an inverse of a).

= a'' (e is an identity of G).

Thus inverse of a is unique in G.

Examples 16:

i) in group (Z, +), inverse of 1 is -1 and inverse of 2 is –2 and so on.

ii) in group $(\mathbb{R} - \{0\}, \times)$ inverse of 3 is $\frac{1}{3}$ etc.

Exercise 2.8

1. Operation $\oplus$ performed on the two-member set $G = \{0,1\}$is shown in the adjoining table. Answer the questions: -

$\oplus$	0	1
0	0	1
1	1	0

i) Name the identity element if it exists?

ii) What is the inverse of 1 ?

iii) Is the set G, under the given operation a group? Abelian or non-Abelian?

2. The operation $\oplus$ as performed on the set {0,1,2,3} is shown in the adjoining table, show that the set is an Abelian group?

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

3. For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.

	<u>Set</u>	<u>Operation</u>
i)	The set of rational numbers	$\times$
ii)	The set of rational numbers	$+$
iii)	The set of positive rational numbers	$\times$
iv)	The set of integers	$+$
v)	The set of integers	$\times$

4. Show that the adjoining table represents the sums of the elements of the set $\{E, O\}$. What is the identity element of this set? Show that this set is an *abelian group*.

$\oplus$	E	O
E	E	O
O	O	E

5. Show that the set $\{1, \omega, \omega^2\}$, when $\omega^3=1$, is an Abelian group w.r.t. ordinary multiplication.

6. If G is a group under the operation and $a, b \in G$, find the solutions of the equations: a ※ $x = b$, x ※ $a = b$

7. Show that the set consisting of elements of the form $a + \sqrt{3}\, b$ (a, b being rational), is an abelian group w.r.t. addition.

8. Determine whether,($P(S)$, ※), where ※ stands for intersection is a semi-group, a monoid or neither. If it is a monoid, specify its identity.

9. Complete the following table to obtain a semi-group under ※

※	a	b	c
a	c	a	b
b	a	b	c
c	-	-	a

10. Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.

CHAPTER

3

Matrices and Determinants

Animation 3.1: Addition of matrix
Source & Credit: elearn.punjab

3.1 Introduction

While solving linear systems of equations, a new notation was introduced to reduce the amount of writing. For this new notation the word matrix was first used by the English mathematician James Sylvester (1814 - 1897). Arthur Cayley (1821 - 1895) developed the theory of matrices and used them in the linear transformations. Now-a-days, matrices are used in high speed computers and also in
other various disciplines.

The concept of determinants was used by Chinese and Japanese but the Japanese mathematician Seki Kowa (1642 - 1708) and the German Mathematician Gottfried Wilhelm Leibniz (1646 - 1716) are credited for the invention of determinants. G. Cramer (1704 - 1752) applied the determinants successfully for solving the systems of linear equations.

A rectangular array o f numbers enclosed by a pair o f brackets such as:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{(i)} \qquad \text{or} \qquad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix} \quad \text{(ii)}$$

is called a **matrix**. The horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. The numbers used in rows or columns are said to be the **entries** or **elements** of the matrix.

The matrix in (i) has two rows and three columns while the matrix in (ii) has 4 rows and three columns. Note that the number of elements of the matrix in (ii) is $4 \times 3 = 12$. Now we give a general definition of a matrix.

Generally, a bracketed rectangular array of $m \times n$ elements
a_{ij} ($i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$), arranged in m rows and n columns such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \qquad \text{(iii)}$$

is called an m by n matrix (written as $m \times n$ matrix).

$m \times n$ is called the order of the matrix in (iii). We usually use capital letters such as A, B, C, X, Y, etc., to represent the matrices and small letters such as $a, b, c, \ldots\ l, m, n, \ldots, a_{11}, a_{12}, a_{13}, \ldots$, etc., to indicate the entries of the matrices.

Let the matrix in (iii) be denoted by A. The ith row and the jth column of A are indicated in the following tabular representation of A.

$$\begin{array}{cc} & \begin{array}{c} \hspace{3em} j\text{th column} \\ \hspace{3em} \downarrow \end{array} \\ \begin{array}{r} \\ \\ \\ A = \\ i\text{th row} \rightarrow \\ \\ \\ \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \text{(iv)}$$

The elements of the ith row o f A are $a_{i1}\ a_{i2}\ a_{i3}\ \ldots\ldots\ a_{ij}\ \ldots\ldots\ a_{in}$
while the elements of the jth column of A are $a_{1j}\ a_{2j}\ a_{3j}\ \ldots\ a_{ij}\ \ldots\ldots\ a_{mj}$.
We note that a_{ij} is the element of the ith row and jth column of A. The double subscripts are useful to name the elements of the matrices. For example, the element 7 is at a_{23} position in

the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$

$A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$, for $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$, where a_{ij} is the element of the ith row and jth column of A.

Note: a_{ij} is also known as the (i, j)th element or entry of A.

The elements (entries) of matrices need not always be numbers but in the study of matrices, we shall take the elements o f the matrices from $\Re$ or from C.

Note: The matrix A is called real if all of its elements are real.

Row Matrix or Row vector: A matrix, which has only one row, i.e., a $1 \times n$ matrix of the form $[a_{i1} \;\; a_{i2} \;\; a_{i3} \;\; \ldots\ldots \;\; a_{in}]$ is said to be a row matrix or a row vector.

Column Matrix or Column Vector: A matrix which has only one column i.e., an $m \times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a column matrix or a column vector.

For example $[1 \;\; -1 \;\; 3 \;\; 4]$ is a row matrix having 4 columns and $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix : If m ≠ n, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular matrix.

For example; $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & 4 \\ 3 & -1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ are rectangular matrices of orders 2×3 and 4×3 respectively.

Square Matrix : If $m = n$, then the matrix of order $m \times n$ is said to be a square matrix of order n or m. i.e., the matrix which has the same number of rows and columns is called a square matrix. For example;

$[\,0\,]$, $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ are square matrices of orders 1, 2 and 3 respectively.

Let $A = [a_{ij}]$ be a square matrix o f order n, then the entries a_{11}, a_{22}, a_{33}, ..., a_{nn} form the principal diagonal for the matrix A and the entries a_{1n}, $a_{2\,n-1}$, $a_{3\,n-2}$, ..., $a_{n-1\,2}$, a_{n1} form the secondary diagonal for the

matrix A. For example, $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, in the matrix the entries of the principal diagonal are a_{11}, a_{22}, a_{33}, a_{44} and the entries of the secondary diagonal are a_{14}, a_{23}, a_{32}, a_{41}.

The principal diagonal of a square matrix is also called the leading diagonal or main diagonal of the matrix.

Diagonal Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a diagonal matrix. The matrices

$[7]$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices.

Scalar Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the matrix A is called a scalar matrix of order n. For example;

$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of order 2, 3 and 4 respectively.

Unit Matrix or Identity Matrix : Let $A = [a_{ij}]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, then the matrix A is called a *unit matrix* or *identity matrix* of order n. We denote such matrix by I_n and it is of the form:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix of order 3 is denoted by I_3, that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix : A square or rectangular matrix whose each element is zero, is called a null or zero matrix. An $m \times n$ matrix with all its elements equal to zero, is denoted by $O_{m\times n}$. Null matrices may be of any order. Here are some examples:

$$[0], [0 \quad 0 \quad 0], \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

O may be used to denote null matrix of any order if there is no confusion.

Equal Matrices: Two matrices of the same order are said to be equal if their corresponding entries are equal. For example, $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are equal, i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \ldots, m$, $j = 1, 2, 3, \ldots, n$. In other words, A and B represent the same matrix.

3.1.1 Addition of Matrices

Two matrices are conformable for addition if they are of the same order. The sum $A + B$ of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $C = [c_{ij}]$ formed by adding the corresponding entries of A and B together. In symbols, we write as $C = A + B$, that is: $[c_{ij}] = [a_{ij} + b_{ij}]$

where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Note that $a_{ij} + b_{ij}$ is the (i, j)th element of $A + B$.

Transpose of a Matrix:

If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A. It is denoted by A^t. If $[a_{ij}]_{m\times n}$ then the transpose of A is defined as:

$A^t = [a'_{ij}]_{n\times m}$ where $a'_{ij} = a_{ji}$ .. for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, m$

For example, if $B = [b_{ij}]_{3\times 4} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then

$B^t = [b'_{ij}]_{4\times 3}$ where $b'_{ij} = b_{ji}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ i.e.,

$$B^t = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \\ b'_{41} & b'_{42} & b'_{43} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$$

Note that the 2nd row of B has the same entries respectively as the 2nd column of B^t and the 3rd row of B^t has the same entries respectively as the 3rd column of B etc.

Example 1:

If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$, then show that

$(A + B)^t = A^t + B^t$

Solution :

$$A + B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+(-1) & -1+3 & 2+1 \\ 3+1 & 1+3 & 2+(-1) & 5+4 \\ 0+3 & -2+1 & 1+2 & 6+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix}$$

and $(A+B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix}$ (1)

Taking transpose of *A* and *B*, we have

$$A^t == \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} \text{ and } B^t \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}, \text{ so}$$

$$A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \quad (2)$$

From (1) and (2), we have $(A+B)^t = A^t + B^t$

3.1.2 Scalar Multiplication

If A = $[a_{ij}]$ is $m \times n$ matrix and *k* is a scalar, then the product of *k* and *A*, denoted by *kA*, is the matrix formed by multiplying each entry of *A* by *k*, that is,

$$kA = [ka_{ij}]$$

Obviously, order of kA is $m \times n$.

Note. If *n* is a positive integer, then $A + A + A + \ldots$ to *n* times = *nA*.

If A = $[a_{ij}] \in M_{m\times n}$ (the set of all $m \times n$ matrices over the real field $\Re$ then $ka_{ij} \in \Re$, for all *i* and *j*, which shows that $kA \in M_{m\times n}$. It follows that the set $M_{m\times n}$ possesses the closure property with respect to scalar multiplication. If $A, B \in M$ and *r*,*s* are scalars, then we can prove that $r(sA) = (rs)A$, $(r+s)A = rA + sA$, $r(A+B) = rA + rB$.

3.1.3 Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then we define subtraction of *B* from *A* as:

$$A - B = A + (-B)$$
$$= [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}] \text{ for } i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n$$

Thus the matrix $A - B$ is formed by subtracting each entry of *B* from the corresponding entry of *A*.

3.1.4 Multiplication of two Matrices

Two matrices *A* and *B* are said to be conformable for the product *AB* if the number of columns of *A* is equal to the number of rows of *B*.

Let $A = [a_{ij}]$ be a 2×3 matrix and $B = [b_{ij}]$ be a 3×2 matrix. Then the product *AB* is defined to be the 2×2 matrix *C* whose element c_{ij} is the sum of products of the corresponding elements of the *i*th row of *A* with elements of *j*th column of *B*. The element c_{21} of ***C*** is shown in the figure (*A*), that is

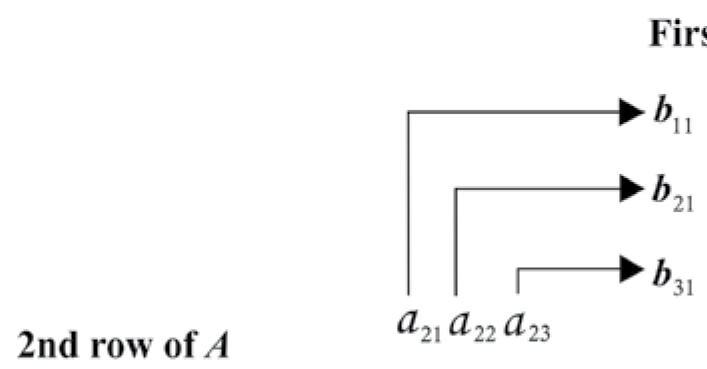


Fig.(A)

$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$. Thus

$$AB == \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Similarly $BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}$$

AB and *BA* are defined and their orders are 2×2 and 3×3 respectively.

Note 1. Both products *AB* and *BA* are defined but $AB \neq BA$

2. If the product AB is defined, then the order of the product can be illustrated as given below:

Order of *A* $\quad m \times n$

Order of *B* $\quad n \times p$

Order of *AB* $\quad m \times p$

Example 2: If $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & 3 \\ -1 & -4 & 6 \\ 0 & -5 & 5 \end{bmatrix}$, then compute A^2B.

Solution :

$$A^2 = AA \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 1 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-1+0 & -2-2+0 & 0+3+0 \\ 2+2-3 & -1+4-6 & 0-6+6 \\ 2+2-2 & -1+4-4 & 0-6+4 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -3 & 0 \\ 2 & -1 & -2 \end{bmatrix}$$

$$\therefore A^2B = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -3 & 0 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 \\ -1 & -4 & 6 \\ 0 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4+0 & -6+16-15 & 9-24+15 \\ 2+3+0 & -2+12+0 & 3-18+0 \\ 4+1+0 & -4+4+10 & 6-6-10 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 0 \\ 5 & 10 & -15 \\ 5 & 10 & -10 \end{bmatrix}$$

Note: Powers of square matrices are defined as:

$A^2 = A \times A$, $A^3 = A \times A \times A$,

$A^n = A \times A \times A \times$ .... to n factors.

3.2 Determinant of a 2 × 2 matrix

We can associate with every square matrix *A* over $\Re$ or *C*, a number $|A|$, known as the determinant of the matrix *A*.

The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets. The number of elements in any row or column is called the order of determinant. For example,

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of *A* is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Its value is defined to be the real number *ad* – *bc*, that is,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example, if $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ and $B \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (2)(3) - (-1)(4) = 6 + 4 = 10$$

and $$|B| = \begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} = (1)(8) - (4)(2) = 8 - 8 = 0$$

Hence the determinant of a matrix is the difference of the products of the entries in the two diagonals.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$-bc \quad ad$

Note: The determinant of a 1×1 matrix $[a_{11}]$ is defined as $|a_{11}| = a_{11}$

3.2.1 Singular and Non-Singular Matrices

A square matrix A is **singular** if $|A| = 0$, otherwise it is a **non singular** matrix. In the above example, $|B| = 0 \Rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ is a singular matrix and $|A| = 10 \neq 0 \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ is a non-singular matrix.

3.2.2 Adjoint of a 2 × 2 Matrix

The adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by *adj A* and is defined as: *adj A* = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3.2.3 Inverse of a 2 × 2 Matrix

Let A be a non-singualr square matrix of **order 2**. If there exists a matrix B such that $AB = BA = I_2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then B is called the multiplicative inverse of A and is usually denoted by A^{-1}, that is, $B = A^{-1}$

$$\boxed{\text{Thus } AA^{-1} = A^{-1}A = I_2}$$

Example 3: For a non-singular matrix A, prove that $A = \dfrac{1}{|A|} adjA$

Solution : If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, Then:

$AA^{-1} = I_2$, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $\begin{bmatrix} ap+br & ap+bs \\ cp+dr & cq+ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{cases} ap+br = 1 \ldots\text{(i)} = \quad ap \quad bs \quad 0 \ldots\text{(ii)} \\ cp+dr = 0 \ldots\text{(iii)} = \quad cq \quad ds \quad 1 \ldots\text{(iv)} \end{cases}$$

From (iii), r = $\dfrac{-c}{d}p$

Putting the value of r in (i), we have

$$ap + b\left(\frac{-c}{d}p\right) = 1 \Rightarrow \left(\frac{ad-bc}{d}\right)p = 1 \Rightarrow p = \frac{d}{ad-bc}$$

and $\quad r = \dfrac{-c}{d}p = \dfrac{-c}{d}\cdot\dfrac{d}{ad-bc} = -\dfrac{c}{ad-bc}$

Similarly, solving (ii) and (iv), we get

$$q = \frac{-b}{ad-bc}, \quad s = \frac{a}{ad-bc}$$

Substituting these values in $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus $A^{-1} = \dfrac{1}{|A|} AdjA$

Example 4: Find A^{-1} if $A=\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}$ and verify that $AA^{-1} = A^{-1}A$

Solution : $|A|=\begin{vmatrix}5 & 3\\ 1 & 1\end{vmatrix} = 5 - 3 = 2$

Since $|A| \neq 0$, we can find A^{-1}.

$$A^{-1}=\frac{1}{|A|}AdjA$$

$$\Rightarrow \quad A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\ \frac{-1}{2} & \frac{5}{2}\end{bmatrix}$$

Now
$$A.A^{-1}=\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\ \frac{-1}{2} & \frac{5}{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{-15}{2}+\frac{15}{2}\\ \frac{1}{2}-\frac{1}{2} & \frac{-3}{2}+\frac{5}{2}\end{bmatrix} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \text{(i)}$$

and
$$A^{-1}.A=\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\ \frac{-1}{2} & \frac{5}{2}\end{bmatrix}\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{3}{2}-\frac{3}{2}\\ \frac{-5}{2}+\frac{5}{2} & \frac{-3}{2}+\frac{5}{2}\end{bmatrix} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii), we have

$$AA^{-1} = A^{-1}A$$

3.3 Solution of simultaneous linear equations by using matrices

Let the system of linear equations be

$$\left.\begin{aligned}a_{11}x_1 + a_{12}x_2 &= b_1\\ a_{21}x_1 + a_{22}x_2 &= b_2\end{aligned}\right\} \qquad \text{(i)}$$

where a_{11}, a_{12}, a_{21}, a_{22}, b_1 and b_1 are real numbers.
The system (i) can be written in the matrix form as:

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix} \quad \text{or} \quad AX = B \qquad \text{(ii)}$$

$$\text{where} \quad A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\end{bmatrix} B=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

If $|A| \neq 0$, then A^{-1} exists so (ii) gives

	$A^{-1}(AX) = A^{-1}B$	(By pre-multiplying (ii) by A^{-1})
or	$(A^{-1}A)X = A^{-1}B$	(Matrix multiplication is associative)
$\Rightarrow$	$X = A^{-1}B$	$(\because A^{-1}A = I_2)$
$\Rightarrow$	$X = A^{-1}B$	

$$\text{or} \quad \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

$$=\frac{1}{|A|}\begin{bmatrix}a_{22}b_1 - a_{12}b_2\\ -a_{21}b_1 - a_{11}b_2\end{bmatrix} = \begin{bmatrix}\frac{b_1a_{22}-a_{12}b_2}{|A|}\\ \frac{a_{11}b_2-b_1a_{21}}{|A|}\end{bmatrix}$$

$$\text{Thus} \quad = \frac{\begin{vmatrix}b_1 & a_{12}\\ b_2 & a_{22}\end{vmatrix}}{|A|} \text{ and } x_2 \ \frac{\begin{vmatrix}a_{11} & b_1\\ b_{21} & b_2\end{vmatrix}}{|A|}$$

It follows from the above discussion that the system of linear equations such as (i) has a unique solution if $|A| \neq 0$.

Example 5: Solve the following systems of linear equations.

i) $\left.\begin{matrix}3x_1 - x_2 = 1\\ x_1 + x_2 = 3\end{matrix}\right\}\begin{matrix}+ \quad =\\ + \quad =\end{matrix}$ ii) $\left.\begin{matrix}x_1 & 2x_2 & 4\\ 2x_1 & 4x_2 & 12\end{matrix}\right\}$

Solution : (i) The matrix form of the system $\left.\begin{matrix}3x_1 - x_2 = 1\\ x_1 + x_2 = 3\end{matrix}\right\}$ is

$$\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}$$

or $AX = B$ . . . . (i) where $A == \begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}$, $X \begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and $B \begin{bmatrix}1\\ 3\end{bmatrix}$

$$|A| = \begin{vmatrix}3 & -1\\ -1 & 1\end{vmatrix} = 3 + 1 = 4$$

and adj $A = \begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}$, therefore,

$$A^{-1} = \frac{1}{4}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4}\end{bmatrix}$$

(I) becomes $X = A^{-1}B$, that is,

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{4} & +\frac{3}{4}\\ -\frac{1}{4} & +\frac{9}{4}\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}$$

$\Rightarrow$ $x_1 = 1$ and $x_2 = 2$

(ii) The matrix form of the system $\left.\begin{matrix}x_1 + 2x_2 = 4\\ 2x_1 + 4x_2 = 12\end{matrix}\right\}$ is

$$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}4\\ 12\end{bmatrix}$$

and $|A| = \begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}$ = 4 – 4 = 0, so A^{-1} does not exist.

Multiplying the first equation o f the above system by 2, we have

$2x_1 + 4x_2 = 8$ but $2x_1 + 4x_2 = 12$

which is impossible. Thus the system has no solution.

Exercise 3.1

1. If $A == \begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$ and $B \begin{bmatrix}1 & 7\\ 6 & 4\end{bmatrix}$, then show that

i) $4A - 3A = A$ ii) $3B - 3A = 3(B - A)$

2. If $A = \begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix}$, show that $A^4 = I_2$.

3. Find x and y if

i) $\begin{bmatrix}x+3 & 1\\ -3 & 3y-4\end{bmatrix} == \begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$ ii) $\begin{bmatrix}x+3 & 1\\ -3 & 3y-4\end{bmatrix} \begin{bmatrix}y & 1\\ -3 & 2x\end{bmatrix}$

4. If $A == \begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}$ and $B \begin{bmatrix}0 & 3 & 2\\ 1 & -1 & 2\end{bmatrix}$, find the following matrices;

i) $4A - 3B = A$ ii) $A + 3(B - A)$

5. Find x and y If $\begin{bmatrix}2 & 0 & x\\ 1 & y & 3\end{bmatrix} + 2\begin{bmatrix}1 & x & y\\ 0 & 2 & -1\end{bmatrix} = \begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$

6. If $A = [a_{ij}]_{3\times3}$, find the following matrices;

i) $\lambda(\mu A)=(\lambda\mu)A$ ii) $(\lambda+\mu)A=\lambda A+\mu A$ ii) $\lambda A-A=(\lambda-1)A$

7. If $A = [a_{ij}]_{2\times3}$ and $B = [b_{ij}]_{2\times3}$, show that $\lambda(A+B)=\lambda A+\lambda B$.

8. If $A=\begin{bmatrix}1 & 2\\ a & b\end{bmatrix}$ and A^2 $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, find the values of a and b.

9. If $A=\begin{bmatrix}1 & -1\\ a & b\end{bmatrix}$ and A^2 $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find the values of a and b.

10. If $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}$ and B $\begin{bmatrix}2 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}$, then show that $(A + B)^t = A^t + B^t$.

11. Find A^3 if $A=\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$

12. Find the matrix X if;.

i) $X\begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}-1 & 5\\ 12 & 3\end{bmatrix}$ ii) $\begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix}X=\begin{bmatrix}2 & 1\\ 5 & 10\end{bmatrix}$

13. Find the matrix A if,

i) $\begin{bmatrix}5 & -1\\ 0 & 0\\ 3 & 1\end{bmatrix}A=\begin{bmatrix}3 & -7\\ 0 & 0\\ 7 & 2\end{bmatrix}$ ii) $\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}A=\begin{bmatrix}0 & -3 & 8\\ 3 & 3 & -7\end{bmatrix}$

14. Show that $\begin{bmatrix}r\cos\phi & 0 & -\sin\phi\\ 0 & r & 0\\ r\sin\phi & 0 & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -r\sin\phi & 0 & r\cos\phi\end{bmatrix}=rI_3$.

3.4 Field

A set F is called a **field** if the operations of addition '+ ' and multiplication '. ' on F satisfy the following properties written in tabular form as:

Addition		**Multiplication**
	Closure	
i) For any $a,b \in F$, $a + b \in F$		For any $a,b \in F$, $a.b \in F$
	Commutativity	
ii) For any $a,b \in F$, $a + b = b + a$		For any $a,b \in F$, $a.b = b.a$
	Associativity	
iii) For any $a,b,c \in F$, $(a + b) + c = a + (b + c)$		For any $a,b,c \in F$, $(a.b).c = a.(b.c)$
	Existence of Identity	
iv) For any $a \in F, \exists\ 0 \in F$ such that $a + 0 = 0 + a = a$		For any $a \in F, \exists\ 1 \in F$ such that $a.1 = 1.a = a$
	Existence of Inverses	
v) For any $a \in F, \exists -a \in F$ such that $a + (-a) = (-a) + a = 0$		v) For any $a \in F$, $a \neq 0$ $\exists \frac{1}{a} \in F$ such that $a.(\frac{1}{a})=(\frac{1}{a}).a=1$
	Distributivity	
vi) For any $a,b,c \in F$,	D_1 : $a(b + c) = ab + ac$ D_2 : $(b + c)a = ba + ca$	

All the above mentioned properties hold for Q, $\Re$, and C.

3.5 Properties of Matrix Addition , Scalar Multiplication and Matrix Multiplication.

If A, B and C are $n\times n$ matrices and c and d are scalers, the following properties are true:

1. **Commutative property w.r.t. addition:** $A + B = B + A$

Note: w.r.t. is used for "with respect to".

2. **Associative property w.r.t. addition:** $(A + B) + C – A + (B + C)$
3. **Associative property of scalar multiplication:** $(cd)A = c(dA)$
4. **Existance of additive identity:** $A + O = O + A – A$ (O is null matrix)
5. **Existance of multiplicative identity:** $IA = AI = A$ (I is unit/identity matrix)
6. **Distributive property w.r.t scalar multiplication:**
   (a) $c(A + B) = cA + cB$ **(b)** $(c + d)A = cA + dA$
7. **Associative property w.r.t. multiplication:** $A(BC) = (AB)C$
8. **Left distributive property:** $A(B + C) = AB + AC$
9. **Right distributive property:** $(A + B)C = AC + BC$
10. $c(AB) = (cA)B = A(cB)$

Example 1: Find AB and BA if $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

Solution : $AB = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+0\times2+1\times1 & 2\times(-1)+0\times3+1\times(-2) & 2\times0+0\times(-1)+1\times3 \\ 1\times1+4\times2+2\times1 & 1\times(-1)+4\times3+2\times(-2) & 1\times0+4\times(-1)+2\times3 \\ 3\times1+0\times2+6\times1 & 3\times(-1)+0\times3+6\times(-2) & 3\times0+0\times(-1)+6\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 3 \\ 11 & 7 & 2 \\ 9 & -15 & 18 \end{bmatrix} \qquad (1)$$

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+(-1)\times1+0\times3 & 1\times0+(-1)\times4+0\times0 & 1\times1+(-1)\times2+0\times6 \\ 2\times2+3\times1+(-1)\times3 & 2\times0+3\times4+(-1)\times0 & 2\times1+3\times2+(-1)\times6 \\ 1\times2+(-2)\times1+3\times3 & 1\times0+(-2)\times4+3\times0 & 1\times1+(-2)\times2+3\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -1 \\ 4 & 12 & 2 \\ 9 & -8 & 15 \end{bmatrix} \qquad (2)$$

Thus from (1) and (2), $AB \neq BA$

Note: Matrix multiplication is not commutative in general

Example 2: If $= \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$,then find AA^t and (A^t).

Solution : Taking transpose of A, we have

$$A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}, \text{ so}$$

$$AA^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9+0 & 2+0+12+0 & -6-5+6+0 \\ 2+0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6+0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

As $A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$, so $\left(A^t\right)^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ which is A,

That is, $(A^t)^t = A$. (Note that this rule holds for any matrix A.)

Exercise 3.2

1. If $A = [a_{ij}]_{3\times4}$, then show that
i) $I_3A = A$ ii) $AI_4 = A$

2. Find the inverses of the following matrices.

i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ iv) $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

3. Solve the following system of linear equations.

i) $\left.\begin{array}{l} 2x_1 - 3x_2 = 5 \\ 5x_1 + x_2 = 4 \end{array}\right\}$ ii) $\left.\begin{array}{l} 4x_1 + 3x_2 = 5 \\ 3x_1 - x_2 = 7 \end{array}\right\}$ iii) $\left.\begin{array}{l} 3x_1 - 5y = 1 \\ -2x + y = -3 \end{array}\right\}$

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -5 \\ -1 & 0 & 4 \end{bmatrix}$, $B \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, then find

i) $A - B$ ii) $B - A$ iii) $(A - B) - C$ iv) $A - (B - C)$

5. If $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ and $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$, then show that

i) $(AB)C = A(BC)$ ii) $(A + B)C = AC + BC$

6. If A and B are square matrices of the same order, then explain why in general;
i) $(A + B)^2 \neq A^2 + 2AB + B^2$ ii) $(A - B)^2 \neq A^2 - 2AB + B^2$
iii) $(A + B)(A - B) \neq A^2 - B^2$

7. If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ then find AA^t and A^tA.

8. Solve the following matrix equations for X:

i) $3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

ii) $2X - 3A = B$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

9. Solve the following matrix equations for A:

(i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$ (ii) $A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

3.6 Determinants

The determinants of square matrices of order $n \geq 3$, can be written by following the same pattern as already discussed. For example, if $n = 4$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \text{ then the determinat of } A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Now our aim is to compute the determinants of various orders. But before describing a method for computation o f determinants of order $n \geq 3$, we introduce the following definitions.

3.6.1 Minor and Cofactor of an Element of a Matrix or its Determinant

Minor of an Element: Let us consider a square matrix A of order 3 .Then the minor of an element a_{ij}, denoted by .M_{ij} is the determinant of the $(3-1)\times(3-1)$ matrix formed by deleting the ith row and the jth column of A(or|A|).

For example, if

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix obtained by deleting the first row and the second column of A is $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ (see adjoining figure) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and its determinant is the minor of an, that is,

$$M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Cofactor of an Element: The cofactor of an element a_{ij} denoted by A_{ij} is defined by $A_{ij}=(-1)^{i+j}\times M_{ij}$

where M_{ij} is the minor of the element a_{ij} of A or $|A|$.

For example, $A_{12}=(-1)^{1+2}M_{12}=(-1)^3\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

3.6.2 Determinant of a Square Matrix of Order $n\geq3$:

The determinant of a square matrix of order n is the sum of the products of each elem ent of row (or column) and its cofactor.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$, then

$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+a_{i3}A_{i3}+\dots+a_{in}A_{in}$ for $i=1,2,3,\dots,n$

or $|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+a_{3j}A_{3j}+\dots+a_{nj}A_{nj}$ for $j=1,2,3,\dots,n$

Putting $i=1$, we have

$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}+\dots+a_{1n}A_{1n}$ which is called the expansion of |A|by the first row.

If A is a matrix of order 3, that is, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+a_{i3}A_{i3}+\dots+a_{in}A_{in}$ for $i=1,2,3$ (1)

or $|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+a_{3j}A_{3j}+\dots+a_{nj}A_{nj}$ for $j=1,2,3$ (2)

For example, for $i=1$, $j=1$ and $j=2$, we have

$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$ (i)

or $|A|=a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31}$ (ii)

or $|A|=a_{12}A_{12}+a_{22}A_{22}+a_{32}A_{32}$ (iii)

(iii) can be written as: $|A|=a_{12}(-1)^{1+2}M_{12}+a_{22}(-1)^{2+2}M_{22}+a_{32}(-1)^{3+2}M_{32}$

i.e., $|A|=-a_{12}M_{12}+a_{22}M_{22}-a_{32}M_{32}$ (iv)

Similarly (i) can be written as $|A|=a_{11}M_{11}-a_{12}M_{12}-a_{13}M_{13}$ (v)

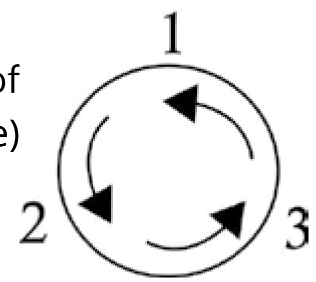
Putting the values of $M_{11}.M_{12}$: and M_{13} in (iv). we obtain

$$|A|=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $|A|=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$ (vi)

or $|A|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$ (vi)'

The second scripts of positive terms are in circular order of anti-clockwise direction i.e., these are as 123, 231, 312 (adjoining figure) while the second scripts of negative terms are such as 132, 213, 321.



An alternative way to remember the expansion of the determinant $|A|$ given in (vi)′ is shown in the figure below.

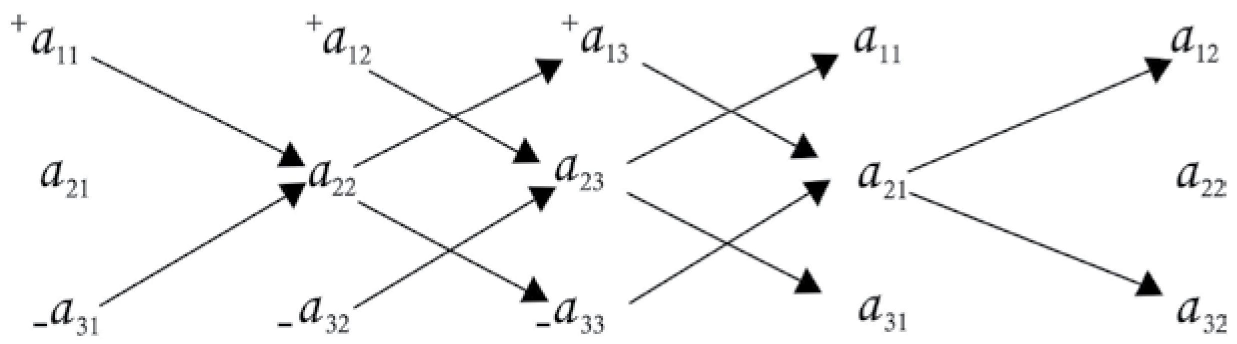


Fig.

Example 1: Evaluate the determinant of $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Solution : $|A| = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{vmatrix}$

Using the result (v) of the Art.3.6.2, that is,

$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, we get,

$$|A| = 1\begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

$= 1[6 - 1(-3)] + 2[(-2).2 - 1.4] + 3[(-2)(-3) - 12]$

$= (6 + 3) + 2(-4 - 4) + 3(6 - 12)$

$= 9 - 16 - 18 = -25$

Example 2: Find the cofactors A_{12}, A_{22} and A_{32} if $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ and find $|A|$.

Solution : We first find M_{12}, M_{22} and M_{32},

$$M_{12} = \begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} = -4 - 4 = -8\,; \quad M_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10 \text{ and } M_{32} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 - (-6) = 7$$

Thus $A_{12} = (-1)^{1+2}M_{12} = (-1)(-8) = 8$; $A_{22} = (-1)^{2+2}M_{22} = 1(-10) = -10$

$A_{32} = (-1)^{3+2}M_{32} = (-1)(7) = -7$;

and $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = (-2)8 + 3(-10) + (-3)(-7)$

$= -16 - 30 + 21 = -25$

Note that $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 1(8) + (-2)(-10) + 4(-7)$

$= 8 + 20 - 28 = 0$

and $a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = 3(8) + 1(-10) + 2(-7)$

$= 24 - 10 - 14 = 0$

Similarly we can show that $a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 0$;

$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$; and $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$;

3.7 Properties of Determinants which Help in their Evaluation

1. For a square matrix A, $|A| = |A^t|$
2. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.
3. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.
4. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.
5. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in \Re$, then the determinant of the resulting matrix is $k\,|A|$.
6. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{then}$$

$$|B|=\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13}\\ a_{21}+b_{21} & a_{22} & a_{23}\\ a_{31}+b_{31} & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix} b_{11} & a_{12} & a_{13}\\ b_{21} & a_{22} & a_{23}\\ b_{31} & a_{32} & a_{33}\end{vmatrix}$$

7. If to each entry of a row (or a column) of a square matrix A is added a non-zero multiple of the corresponding entry of another row (or column), then the determinant of the resulting matrix is $|A|$.
8. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Now we prove the above mentioned properties of determinants.

Property 1: If the rows and columns of a determinant are interchanged, then the value of the determinant does not change. For example.

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{vmatrix}$$ (rows and columns are interchanged)

Property 2: The value of a determinant changes sign if any two rows (columns) are interchanged. For example,

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and $\begin{vmatrix} a_{12} & a_{11}\\ a_{22} & a_{21}\end{vmatrix} = a_{12}a_{21} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21})$ (columns are interchanged)

Property 3: If all the entries in any row (column) are zero, the value of the determinant is zero. For example,

$$\begin{vmatrix} 0 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{vmatrix} = 0\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - 0\begin{vmatrix} a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix} + 0\begin{vmatrix} a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix} = 0$$ (expanding by C)

Property 4: If any two rows (columns) of a determinant are identical, the value of the determinant is zero. For example,

$$\begin{vmatrix} a & b & c\\ a & b & c\\ x & y & z\end{vmatrix} = 0,$$ (it can be proved by expanding the determinant)

Property 5: If any row (column) of a determinant is multiplied by a non-zero number k, the value of the new determinant becomes equal to k times the value of original determinant. For example,

$|A| = \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$, multiplying first row by a non-zero number k, we get

$$\begin{vmatrix} ka_{11} & ka_{12}\\ a_{21} & a_{22}\end{vmatrix} = ka_{11}a_{22} - ka_{12}a_{21} = k(a_{11}a_{22} - a_{12}a_{21}) = k\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$

Property 6: If any row (column) of a determinant consists of two terms, it can be written as the sum of two determinants as given below:

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13}\\ a_{21}+b_{21} & a_{22} & a_{23}\\ a_{31}+b_{31} & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix} b_{11} & a_{12} & a_{13}\\ b_{21} & a_{22} & a_{23}\\ b_{31} & a_{32} & a_{33}\end{vmatrix}$$ (proof is left for the reader)

Property 7: If any row (column) o f a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change. For example,

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = \begin{vmatrix} a_{11} & a_{12}+ka_{11}\\ a_{21} & a_{22}+ka_{21}\end{vmatrix}$$ (k multiple of C_1 is added to C_2)

It can be proved by expanding both the sides. Proof is left for the reader.

Example 3: If $A = \begin{bmatrix} 2 & -2 & 3 & 4\\ 3 & 1 & 5 & -1\\ -5 & -3 & 1 & 0\\ 1 & -1 & 0 & 2\end{bmatrix}$, evaluate $|A|$

Solution : $|A| = \begin{vmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 3 & 0 \\ 0 & 4 & 5 & -7 \\ 0 & -8 & 1 & 10 \\ 1 & -1 & 0 & 2 \end{vmatrix}$ By $R_1 + (-2)R_4$, $R_2 + (-3)R_4$ and $R_3 + 5R_4$

Expanding by first column, we have

$|A| = 0.A_{11} + 0.A_{21} + 0.A_{31} + 1.A_{41}$

$= (-1)^{4+1} \times \begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix}$

$= (-1)(-3)[4 \times 10 - (-7)(-8)] = 3(40 - 56) = -48$

Example 4: Without expansion, show that $\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = 0$

Solution : Multiplying each entry of C_1 by -1 and adding to the corresponding entry of C_2 i.e., by $C_2 + (-1)C_1$, we get

$$\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = \begin{vmatrix} x & a+x+(-1)x & b+c \\ x & b+x+(-1)x & c+a \\ x & c+x+(-1)x & a+b \end{vmatrix}$$

$$= \begin{vmatrix} x & a & b+c \\ x & b & c+a \\ x & c & a+b \end{vmatrix} = x\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} \left(\begin{array}{c}\text{by property 5 or} \\ \text{taking x common} \\ \text{from } C_1\end{array}\right)$$

$$= x\begin{vmatrix} 1 & a+(b+c) & b+c \\ 1 & b+(c+a) & c+a \\ 1 & c+(a+b) & a+b \end{vmatrix}, \left(\begin{array}{c}\text{adding the entries of } C_3 \text{ to the} \\ \text{corresponding entries of } C_2\end{array}\right)$$

$= x(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$, (by property 5)

$= x(a+b+c).0$ ($\because$ C_1 and C_2 are identical or by property 3)

$= 0$

Example 5: Solve the equation $\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -2 & 3 & 4 \\ -2 & x & 1 & -1 \end{vmatrix} = 0$

Solution: By $C_3 + C_2$ and $C_4 + C_2$, we have

$$\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 2 \\ -2 & x & x+1 & x-1 \end{vmatrix} = 0$$

Expanding by R_2, we get $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ -2 & x+1 & x-1 \end{vmatrix} = 0$ ($\because$ $(-1)^{2+2} = 1$)

By $R_3 + 2R_2$, we get $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & x+3 & x+3 \end{vmatrix} = 0$

or $(x+3)\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$ (by taking $x + 3$ common from R_3)

$\Rightarrow$ $x + 3 = 0$ or $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $x = -3$ or $x = 0$ ($\because$ R_1 and R_3 are identical if $x = 0$)

Thus the solution set is $\{-3, 0\}$.

3.8 Adjoint and Inverse of a Square Matrix of Order $n \geq 3$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix of co-factors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$,

and $\text{adj}\, A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

Inverse of a Square Matrix of Order n≥3: Let A be a non singular square matrix of order n. If there exists a matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.

Thus $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

If A is a non singular matrix, then

$$A^{-1} = \frac{1}{|A|}\text{adj}A$$

Example 6: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution: We first find the cofactors of the elements of A.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 1.(2+1) = 3, \qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1) = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 1.(0-2) = -2, \qquad A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = (-1)(0+2) = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1.(1-2) = -1, \qquad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = (-1)(-1-0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 1.(0-4) = -4, \qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1-0) = -1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1.(2-0) = 2$$

Thus $[A_{ij}]_{3\times3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

and $\text{adj}\, A = [A'_{ij}]_{3\times3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ $(\because A'_{ij} = A_{ji}$ for $i, j = 1, 2, 3)$

Since $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
$= 1(3) + 0(1) + 2(-2)$
$= 3 + 0 - 4 = -1,$

So $A^{-1} = \frac{1}{|A|}\text{adj}\, A = \frac{1}{-1}\begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$

Example 7: If $A = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ then verify that

$(AB)^t = B^tA^t$

Solution: so $AB = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1-4 & -3+2 \\ 1-8 & 3+4 \\ 2+2 & 6-1 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -7 & 7 \\ 4 & 5 \end{bmatrix}$,

$(AB)^t = \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$

and $B^tA^t = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \left(\because A^t = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \text{and } B^t = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\right)$

$= \begin{bmatrix} -1-4 & 1-8 & 2+2 \\ -3+2 & 3+4 & 6-1 \end{bmatrix} \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$

Thus $(AB)^t = B^tA^t$

Exercise 3.3

Evaluate the following determinants.

1. i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$ v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

2. Without expansion show that

i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$ 0 iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ 0

3. Show that

i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ iii) $\begin{vmatrix} a+l & a & a \\ a & a\ l+ & a \\ a & a & a+l \end{vmatrix}$ $l^2(3a\ \ l+)$

iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & +z \\ x^2 & y^2 & z^2 \end{vmatrix}$ v) $\begin{vmatrix} b+c & a & a \\ =b & c\ \ a & b \\ c & c & a+b \end{vmatrix}$ $4abc$

vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3\ \ b^3$ vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix}$ r

viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

4. If If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$, then find;

i) A_{12}, A_{22}, A_{32} and $|A|$ ii) B_{21}, B_{22}, B_{23} and $|B|$

5. Without expansion verify that

i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$ ii) $= \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$ 0 iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix}$ 0

iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$

viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$ ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$

6. Find values of x if

i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = 30$ ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$ iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

7. Evaluate the following determinants:

i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

8. Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

9. Find $|AA^t|$ and $|A^tA|$ if

i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

10. If A is a square matrix of order 3, then show that $|KA| = k^3|A|$.

11. Find the value of λ if A and B singular.

$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$

12. Which of the following matrices are singular and which of them are non singular?

i) $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$

13. Find the inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ and show that $A^{-1}A = I_3$

14. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ if

i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

15. Verify that $(AB)^t = B^tA^t$ and if

$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

16. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ verify that $(A^{-1})^t = (A^t)^{-1}$

17. If A and B are non-singular matrices, then show that

i) $(AB)^{-1} = B^{-1}A^{-1}$ ii) $(A^{-1})^{-1} = A$

3.9 Elementary Row and Column Operations on a Matrix

Usually a given system of linear equations is reduced to a simple equivalent system by applying in turn a finite number of elementary operations which are stated as below:

1. Interchanging two equations.
2. Multiplying an equation by a non-zero number.
3. Adding a multiple of one equation to another equation.

Note: The systems of linear equations involving the same variables, are equivalent if they have the same solution.

Corresponding to these three elementary operations, the following elementary row operations are applied to matrices to obtain equivalent matrices.

i) Interchanging two rows
ii) Multiplying a row by a non-zero number
iii) Adding a multiple of one row to another row.

Note: Matrices A and B are equivalent if B can be obtained by applying in turn a finite number of row operations on A.

Notations that are used to represent row operations for I to III are given below:
Interchanging R_i and R_j is expressed as $R_i \leftrightarrow R_j$.
k times R_i, is denoted by $kR_i \to R'_i$
Adding k times R_j to R_i is expressed as $R_i + kR_j \to R'_i$
(R'_i is the new row obtained after applying the row operation).

For equivalent matrices A and B, we write $A \underset{\sim}{R} .B$.

If $A \underset{\sim}{R} B$ then $B \underset{\sim}{R} A$. Also if $A \underset{\sim}{R} B$ and $B \underset{\sim}{R} C$, then $A \underset{\sim}{R} C$. Now we state the elementary column operations and notations that are used for them.

i) Interchanging two columns $C_i \leftrightarrow C_j$
ii) Multiplying a column by a non-zero number $kC_i \to C'_i$
iii) Adding a multiple of one column to another column $C_i + kC_j \to C'_i$

Consider the system of linear equations;

$$\left.\begin{aligned} x+y+2z&=1\\ 2x-y=8z&=12\\ 3x+5y+4z&=\ \ 3 \end{aligned}\right\}$$ which can be written in matrix forms as

$$\begin{bmatrix} 1 & 1 & 2\\ 2 & -1 & 8\\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 1\\ 12\\ -3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 12 & -3 \end{bmatrix}$$

that is, $AX = B$ (i) $X^t A^t = B^t$ (ii)

where $A=\begin{bmatrix} 1 & 1 & 2\\ 2 & 1 & 8\\ 3 & 5 & 4 \end{bmatrix}, X \begin{bmatrix} x\\ y\\ z \end{bmatrix}$ and $B \begin{bmatrix} 1\\ 12\\ -3 \end{bmatrix}$

A is called the matrix of coefficients.

Appending a column of constants on the left of A, we get the augmented matrix of the given system, that is,

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1\\ 2 & -1 & 8 & 12\\ 3 & 5 & 4 & -3 \end{array}\right]$$ (Appended column is separated by a dotted line segment)

Now we explain the application of elementary operations on the system-of linear equations and the application of elementary row operations on the augmented matrix of the system writing them side by side.

$$\left.\begin{aligned} x+y+2z&=1\\ 2x+-y+8z&=12\\ 3x+5y+4z&=\ \ 3 \end{aligned}\right\} \qquad \left[\begin{array}{ccc:c} 1 & 1 & 2 & 1\\ 2 & 1 & 8 & 12\\ 3 & 5 & 4 & -3 \end{array}\right]$$

Adding –2 times the first equation to the second and –3 times the first equation to the third, we get (By $R_2 + (-2)R_1 \to R'_2$ and $R_3 + (-3)R_1 \to R'_3$, we get)

$$\left.\begin{aligned} x+y+2z&=1\\ -3y+4z&=10\\ 2y-2z&=\ \ 6 \end{aligned}\right\} \qquad \underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1\\ 0 & 3 & 4 & 10\\ 0 & 2 & -2 & -6 \end{array}\right]$$

Interchanging the second and third equations, we have (By $R_2 \leftrightarrow R_3$, we get)

$$\left.\begin{aligned} x+y+2z&=1\\ 2y-2z&=6\\ -3y+4z&=10 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 2 & -2 & \vdots & -6\\ 0 & 3 & 4 & \vdots & 10 \end{bmatrix}$$

Multiplying the second equation by $\frac{1}{2}$, we get By $\frac{1}{2}R_2 \to R'_2$, we get.

$$\left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ -3y+4z&=10 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & -3 & 4 & \vdots & 10 \end{bmatrix}$$

Adding 3 times the second equation to By $R_3 + 3R_2 \to R'_3$, we obtain, the third, we obtain,

$$\left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ z&=1 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

The given system is reduced to the triangular form which is so called because on the left the coefficients (of the terms) within the dotted triangle are zero.

Putting $z = 1$ in $y - z = -3$, we have $y - 1 = -3 \Rightarrow y = -2$

Substiliting $z = 1$, $y = -2$ in the first equation, we get

$x + (-2) + 2(1) = 1 \Rightarrow x = 1$

Thus the solution set o f the given system is $\{(1, -2, 1)\}$.

Appending a row of constants below the matrix A^t, we obtain the

augmented matrix for the matrix equation (ii), that is $\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4\\ \cdots & \cdots & \cdots\\ 1 & 12 & -3 \end{bmatrix}$

Now we apply elementary column operations to this augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4\\ \cdots & \cdots & \cdots\\ 1 & 12 & -3 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0\\ 1 & -3 & 2\\ 2 & 4 & -2\\ \cdots & \cdots & \cdots\\ 1 & 10 & -6 \end{bmatrix} \begin{aligned} &\text{By } C_2 + (-2)C_1 \to C'_2 \text{ and}\\ &C_3 + (-3)C_1 \to C'_3 \end{aligned}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 2 & -3\\ 2 & -2 & 4\\ \cdots & \cdots & \cdots\\ 1 & -6 & 10 \end{bmatrix} \text{By } C_2 \leftrightarrow C_3 \qquad \underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & -3\\ 2 & -1 & 4\\ \cdots & \cdots & \cdots\\ 1 & -3 & 10 \end{bmatrix} \text{By } \frac{1}{2}C_2 \to C'_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1 & 1\\ \cdots & \cdots & \cdots\\ 1 & -3 & 1 \end{bmatrix} \text{By } C_3 + 3C_2 \to C'_3$$

Thus $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}$

or $\begin{bmatrix} x+y+2z & y-z & z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}$

$$\Rightarrow \quad \left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ z&=1 \end{aligned}\right\}$$

Upper Triangular Matrix: A square matrix $A = [a_{ij}]$ is called upper triangular if all elements below the principal diagonal are zero, that is,

$a_{ij} = 0$ for all $i > j$

Lower Triangular Matrix: A square matrix $A = [a_{ij}]$ is said to be lower triangular if all elements above the principal diagonal are zero, that is,

$a_{ij} = 0$ for all $i < j$

Triangular Matrix: A square matrix A is named as triangular whether it is upper triangular or lower triangular. For example, the matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 1 & 5 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix}$$ are triangular matrices of order 3 and 4

respectively. The first matrix is upper triangular while the second is lower triangular.

Note: Diagonal matrices are both upper triangular and lower triangular.

Symmetric Matrix: A square matrices $A = [a_{ij}]_{n \times n}$ is called symmetric if $A^t = A$.
From $A^t = A$, it follows that $[a'_{ij}]_{n \times n} = [a_{ij}]_{n \times n}$
which implies that $a'_{ij} = a_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.
but by the definition of transpose, $a'_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.
Thus $a_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.
and we conclude that a square matrix $A = [a_{ij}]_{n \times n}$ is symmetric if $a_{ij} = a_{ji}$.
For example, the matrices

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 2 & -1 \\ 3 & 0 & 5 & 6 \\ 2 & 5 & 1 & -2 \\ -1 & 6 & -2 & 3 \end{bmatrix}$$ are symmetric.

Skew Symmetric Matrix : A square matrix A = $[a_{ij}]_{n \times n}$ is called skew symmetric or anti-symmetric if $A^t = -A$.

From $A^t = -A$, it follows that $[a'_{ij}]$ = for $i, j = 1, 2, 3, \ldots\ldots, n$
which implies that $a'_{ij} = -a_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$
but by the definition of transpose $a'_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots, n$
Thus $-a_{ij} = a_{ji}$ or $a_{ij} = -a_{ji}$

Alternatively we can say that a square matrix $A = [a_{ij}]_{nxn}$ is anti-symmetric if $a_{ij} = -a_{ji}$.

For diagonal elements $j = i$, so
$a_{ii} = -a_{ii}$ or $2a_{ii} = 0 \Rightarrow$ $a_{ii} = 0$ for $i = 1, 2, 3, \ldots\ldots, n$

For example if B = $\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$,then

$$B^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} = -B$$

Thus the matrix B is skew-symmetric.

Let $A = [a_{ij}]$ be an n x m matrix with complex entries, Then the n x m matrix $[\bar{a}_{ij}]$ where $\bar{a}_{ij}$ is the complex conjugate of a_{ij} for all i, j, is called *conjugate* of A and is denoted by $\bar{A}$. For example, if

$$A = \begin{bmatrix} 3-i & -i \\ 2i & 1+i \end{bmatrix}, \text{then } \bar{A} \quad \begin{bmatrix} \overline{3-i} & \overline{-i} \\ \overline{2i} & \overline{1+i} \end{bmatrix} \begin{bmatrix} 3+i & i \\ -2i & 1-i \end{bmatrix}$$

Hermitian Matrix: A square matrix $A = [a_{ij}]_{nxn}$ with complex entries, is called *hermitian* if $\left(\bar{A}\right)^t = A$.

From, $\left(\bar{A}\right)^t = A$ it follows that $\left[\bar{a}'_{ij}\right]_{n \times n} = [a_{ij}]_{nxn}$ which implies that $\bar{a}'_{ij} = a_{ij}$ for $i, j = 1, 2, 3, \ldots, n$ but by the definition of transpose, $\bar{a}'_{ij} = \bar{a}_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.

Thus $a_{ij} = \bar{a}_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$ and we can say that a square matrix $A = [a_{ij}]_{nxn}$ is *hermitian* if $a_{ij} = \bar{a}_{ji}$ for $i, j = 1, 2, 3, \ldots, n$.

For diagnal elements, $j = i$ so $a_{ii} = \bar{a}_{ii}$ which implies that a_{ii} is real for $i = 1, 2, 3, \ldots, n$

For example, if A = $\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$, then

$$\bar{A} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \Rightarrow \left(\bar{A}\right)^t = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} = A$$

Thus A is hermitian.

Skew Hermitian Matrix: A square matrix $A = [a_{ij}]_{nxn}$ with complex entries, is called skew-hermitian or anti-hermitian if $\left(\overline{A}\right)^t = -A$.

From $\left(\overline{A}\right)^t = -A$, it follows that $\left[\overline{a}_{ij}^{/}\right]_{n\times n} = [-a_{ij}]_{nxn}$

which implies that $\overline{a}_{ij}^{/} = -a_{ij}$ for $i, j = 1, 2, 3, ..., n$.

but by the definition of transpose, $\overline{a}_{ij}^{/} = \overline{a}_{ji}$ for $i, j = 1, 2, 3, ..., n$.

Thus $-a_{ij} = \overline{a}_{ji}$ or $a_{ij} = -\overline{a}_{ji}$, for $i, j = 1, 2, 3, ..., n$.

and we can conclude that a square matrix $A = [a_{ij}]_{nxn}$ is anti-hermitian if $a_{ij} = -\overline{a}_{ji}$

For diagonal elements $j = i$, so $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + -\overline{a}_{ii} = 0$

which holds if $a_{ii} = 0$ or $a_{ii} = i\lambda$ where λ is real

because $0 + 0 = 0$ or $i\lambda + i\overline{\lambda} = i\lambda - i\lambda = 0$

For example, if $A = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$, then

$$\overline{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$\Rightarrow \left(\overline{A}\right)^t = \begin{bmatrix} 0 & -2+3i \\ 2+3i & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 2-3i \\ -2-3i & 0 \end{bmatrix} = -A$$

Thus A is skew-hermitian.

3.10 Echelon and Reduced Echelon Forms of Matrices

In any non-zero row of a matrix, the first non-zero entry is called the *leading entry* of that row. The zeros before the leading entry of a row are named as the *leading zero entries* of the row.

Echelon Form of a Matrix: An m x n matrix A is called in **(row) echelon form** if

i) In each successive non-zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row,

ii) Every non-zero row in A precedes every zero row (if any),

iii) The first non-zero entry (or leading entry) in each row is 1.

Note: Some authors do not require the condition (iii)

The matrices $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form

but the matrices $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ are not in echelon form.

Reduced Echelon Form of a Matrix: An m x n matrix A is said to be in **reduced (row) echelon form** if it is in (row) echelon form and if the first non-zero entry (or leading entry) in R_i lies in C_j, then all other entries of C_j are zero.

The matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Example 1: Reduce the following matrix to (row) echelon and reduced (row) echelon form,

$$\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$$

Solution: $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ $\quad R\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & -1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ By $R_1 \leftrightarrow R_2$

$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -5 & 15 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $R_2+(-2)R_1 \to R'_2$ and $R_3+(-3)R_1 \to R'_3$ $\quad \underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $\frac{1}{5}R_2 \to R'_2$

$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}$ By $R_3 +(-4)R_2 \to R'_3$ $\quad \underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ By $R_1 +1.R_2 \to R'_1$

$$\underset{\sim}{R}\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{bmatrix}\quad \begin{array}{l}\text{By } R_1+(-1)R_3\to R'_1\\ \text{and } R_2+1.R_3\to R'_2\end{array}$$

Thus $\begin{bmatrix}1&-1&2&3\\0&1&-1&3\\0&0&1&-1\end{bmatrix}$ and $\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{bmatrix}$ are (row) echelon and reduced (row) echelon forms of the given matrix respectively.

Let A be a non-singular matrix. If the application of elementary row operations on $A \vdots I$ in succession reduces A to I, then the resulting matrix is $I \vdots A^{-1}$.

Similarly if the application of elementary column operations on $\begin{matrix}A\\ \cdots\\ I\end{matrix}$ in succession reduces A to I, then the resulting matrix is $\dfrac{I}{A^{-1}}$

Thus $A \vdots I \ \underset{\sim}{R}\ I \vdots A^{-1}$ and $\begin{matrix}A\\ \cdots\\ I\end{matrix}\ \underset{\sim}{C}\ \begin{matrix}I\\ \cdots\\ A^{-1}\end{matrix}$

Example 2: Find the inverse of the matrix A = $\begin{bmatrix}2&5&-1\\3&4&2\\1&2&-2\end{bmatrix}$

Solution: $|A| = \begin{vmatrix}2&5&-1\\3&4&2\\1&2&-2\end{vmatrix}$ = 2(–8 – 4) – 5(–6 – 2) –1(6 – 4) = –24 + 40 – 2

= 40 – 26 = 14 As $|A| \neq 0$, so A is non-singular.

Appending I_3 on the left of the matrix A, we have $\left[\begin{array}{ccc|ccc}2&5&-1&1&0&0\\3&4&2&0&1&0\\1&2&-2&0&0&1\end{array}\right]$

Interchanging R_1 and R_3 we get..

$$\left[\begin{array}{ccc|ccc}1&2&-2&0&0&1\\3&4&2&0&1&0\\2&5&-1&1&0&0\end{array}\right]\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1&2&-2&0&0&1\\0&-2&8&0&1&-3\\0&1&3&1&0&-2\end{array}\right]\begin{array}{l}\text{By } R_2+(-3)R_1\to R'_2\\ \text{and } R_3+(-2)R_1\to R'_3\end{array}$$

By $-\frac{1}{2}R_2\to R'_2$, we get

$$\left[\begin{array}{ccc|ccc}1&2&-2&0&0&1\\0&1&-4&0&-\frac{1}{2}&\frac{3}{2}\\0&1&3&1&0&-2\end{array}\right]\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1&0&6&0&0&-2\\0&1&-4&0&-\frac{1}{2}&\frac{3}{2}\\0&0&7&1&\frac{1}{2}&-\frac{7}{2}\end{array}\right]\begin{array}{l}\text{By } R_3+(-1)R_2\to R'_3\\ \text{and } R_1+(-2)R_2\to R'_1\end{array}$$

By $\frac{1}{7}R_3\to R'_3$, we have

$$\left[\begin{array}{ccc|ccc}1&0&6&0&1&-2\\0&1&-4&0&-\frac{1}{2}&\frac{3}{2}\\0&0&1&\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{array}\right]\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1&0&0&-\frac{6}{7}&\frac{4}{7}&1\\0&1&0&\frac{4}{7}&-\frac{3}{14}&-\frac{1}{2}\\0&0&1&\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{array}\right]\begin{array}{l}\text{By } R_1+(-6)R_3\to R'_1\\ \text{and } R_2+4R_3\to R'_2\end{array}$$

Thus the inverse of A is $\begin{bmatrix}-\frac{6}{7}&\frac{4}{7}&1\\-\frac{4}{7}&-\frac{3}{14}&-\frac{1}{2}\\\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{bmatrix}$

Appending I_3 below the matrices A, we have

$$\begin{bmatrix}2&5&-1\\3&4&2\\1&2&-2\\ \cdots&\cdots&\cdots\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Interchanging C_1 and C_3, we get

$$\left[\begin{array}{ccc}2&5&-1\\3&4&2\\1&2&-2\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\underset{\sim}{C}\left[\begin{array}{ccc}-1&5&2\\2&4&3\\-2&2&1\\\hline 0&0&1\\0&1&0\\1&0&0\end{array}\right]\underset{\sim}{C}\left[\begin{array}{ccc}1&5&2\\-2&4&3\\2&2&1\\\hline 0&0&1\\0&1&0\\-1&0&0\end{array}\right]\text{ By } (-1)C_1 \to C'_1$$

By $C_2 + (-5)C_1 \to C'_2$ and $C_3 + (-2)C_1 \to C'_3$, we have

$$\left[\begin{array}{ccc}1&0&0\\-2&14&7\\2&-8&-3\\\hline 0&0&1\\0&1&0\\-1&5&2\end{array}\right]\underset{\sim}{C}\left[\begin{array}{ccc}1&0&0\\-2&1&7\\2&-\frac{4}{7}&-3\\\hline 0&0&1\\0&\frac{1}{14}&0\\-1&\frac{5}{14}&2\end{array}\right]\quad\text{By } \frac{1}{14}C_2 \to C'_2$$

By $C_1 + (2)C_2 \to C'_1$ and $C_3 + (-7)C_2 \to C'_3$ we have

$$\left[\begin{array}{ccc}1&0&0\\0&1&0\\\frac{6}{7}&-\frac{4}{7}&1\\\hline 0&0&1\\\frac{1}{7}&\frac{1}{14}&\frac{1}{2}\\-\frac{2}{7}&\frac{5}{14}&-\frac{1}{2}\end{array}\right]\underset{\sim}{C}\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\\hline -\frac{6}{7}&\frac{4}{7}&1\\\frac{4}{7}&-\frac{3}{14}&-\frac{1}{2}\\\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{array}\right]\begin{array}{l}\text{By } C_1 + \left(-\frac{6}{7}\right)C_3 \to C'_1\\ \text{and } C_2 + \left(\frac{4}{7}\right)C_3 \to C'_2\end{array}$$

Thus the inverse of A is $\begin{bmatrix}-\frac{6}{7}&\frac{4}{7}&1\\\frac{4}{7}&-\frac{3}{14}&-\frac{1}{2}\\\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{bmatrix}$

Rank of a Matrix: Let A be a non-zero matrix. If r is the number of non-zero rows when it is reduced to the reduced echelon form, then r is called the (row) **rank** of the matrix A.

Example 3: Find the rank of the matrix $\begin{bmatrix}1&-1&2&-3\\2&0&7&-7\\3&1&12&-11\end{bmatrix}$

Solution: $\begin{bmatrix}1&-1&2&-3\\2&0&7&-7\\3&1&12&-11\end{bmatrix}\underset{\sim}{R}\begin{bmatrix}1&-1&2&-3\\0&2&3&-1\\0&4&6&-2\end{bmatrix}\begin{array}{l}\text{By } R_2+(-2)R_1\to R'_2\\\text{and } R_3+(-3)R_1\to R'_3\end{array}$

$$\underset{\sim}{R}\begin{bmatrix}1&-1&2&-3\\0&2&\frac{3}{2}&-\frac{1}{2}\\0&4&6&-2\end{bmatrix}\text{By } \frac{1}{2}R_2\to R'_2\ \underset{\sim}{R}\begin{bmatrix}1&-1&2&-3\\0&1&\frac{3}{2}&-\frac{1}{2}\\0&0&0&0\end{bmatrix}\text{By } R_3+(-4)R_2\to R'_3$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&\frac{7}{2}&-\frac{7}{2}\\0&1&\frac{3}{2}&-\frac{1}{2}\\0&0&0&0\end{bmatrix}\text{By } R_1+1.R_2\to R'_1$$

As the number of non-zero rows is 2 when the given matrix is reduced to the reduced echelon form, therefore, the rank of the given matrix is 2.

Exercise 3.4

1. If $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$, then show that $A + B$ is symmetric.

2. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$, show that

i) $A + A^t$ is symmetric ii) $A - A^t$ is skew-symmetric.

3. If A is any square matrix of order 3, show that

i) $A + A^t$ is symmetric and ii) $A - A^t$ is skew-symmetric.

4. If the matrices A and B are symmetric and $AB = BA$, show that AB is symmetric.

5. Show that AA^t and A^tA are symmetric for any matrix of order 2 x 3.

6. If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, show that

i) $A + \left(\overline{A}\right)^t$ is hermitian ii) $A - \left(\overline{A}\right)^t$ is skew-hermitian.

7. If A is symmetric or skew-symmetric, show that A^2 is symmetric.

8. If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, find $A\left(\overline{A}\right)^t$.

9. Find the inverses of the following matrices. Also find their inverses by using row and column operations.

i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

10. Find the rank of the following matrices

i) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

3.11 System of Linear Equations

An equation of the form:

$$ax + by = k \qquad \text{(i)}$$

where $a \neq 0$, $b \neq 0$, $k \neq 0$

is called a non-homogeneous linear equation in two variables x and y.

Two linear equations in the same two variables such as:

$$\left.\begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned}\right\} \qquad \text{(I)}$$

is called a system of non-homogeneous linear equations in the two variables x and y if constant terms k_1, k_2 are not both zero.

If in the equation (i), $k = 0$, that is, $ax + by = 0$, then it is called a homogeneous linear equation in x and y.

If in the system (I), $k_1 = k_2 = 0$, then it is said to be a system of homogenous linear equations in x and y.

An equation of the form:

$$ax + by + cz = k \qquad \text{.... (ii)}$$

is called a non-homogeneous linear equation in three variables x, y and z if $a \neq 0$, $b \neq 0$, $c \neq 0$ and $k \neq 0$. Three linear equations in three variables such as:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}\right\} \qquad \text{(II)}$$

is called a system of non-homogeneous linear equations in the three variables x, y and z, if constant terms k_1, k_2 and k_3 are not all zero.

If in the equations (ii) $k = 0$ that is, $ax + by + cz = 0$.
then it is called a homogeneous linear equation it x, y and z.

If in the system (II), $k_1 = k_2 = k_3 = 0$, then it is said to be a system of homogeneous linear equations in x ; y and z.

A system of linear equations is said to be consistent if the system has a unique solution or it has infinitely many solutions.

A system of linear equations is said to be inconsistent if the system has no solution.

The system (II), consists of three equations in three variables so it is called 3 x 3 linear system but a system of the form:

$$\left.\begin{array}{l} x - y + 2z = 6 \\ 2x + y + 3z = 4 \end{array}\right\}$$

is named as 2 x 3 linear system.
Now we solve the following three 3 x 3 linear systems to determine the criterion for a system to be consistent or for a system to be inconsistent.

$$\left.\begin{array}{l} 2x + 5y - z = 5 \\ 3x + 4y + 2z = 11 \\ x + 2y - 2z = 3 \end{array}\right\} \quad ....(1), \qquad \left.\begin{array}{l} x + y + 2z = 1 \\ 2x - y + 7z = 11 \\ 3x + 5y + 4z = 3 \end{array}\right\} \quad (2)$$

and $$\left.\begin{array}{l} x - y + 2z = 1 \\ 2x - 6y + 5z = 7 \\ 3x + 5y + 4z = 3 \end{array}\right\} \quad ....(3)$$

The augmented matrix of the system (1) is

$$\left[\begin{array}{ccccc} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{array}\right]$$

We apply the elementary row operations to the above matrix to reduce it to the equivalent reduced (row) echelon form, that is,

$$\left[\begin{array}{ccccc} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 2 & -2 & \vdots & -3 \\ 3 & 4 & 2 & \vdots & 11 \\ 2 & 5 & -1 & \vdots & 5 \end{array}\right] \text{ BY } R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 2 & 5 & -1 & \vdots & 5 \end{array}\right] \text{BY } R_2 + (-3)R_1 \to R_2' \ \underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 0 & 1 & 3 & \vdots & 11 \end{array}\right] \text{BY } R_3 + (-2)R_1 \to R_3'$$

$\text{BY } -\dfrac{1}{2}R_2 \to R_2',$ we get

$$\left[\begin{array}{ccccc} 1 & 2 & -2 & \vdots & -3 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 1 & 3 & \vdots & 11 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 7 & \vdots & 21 \end{array}\right] \begin{array}{l} \text{By } R_1 + (-2)R_2 \to R'_1 \\ \text{and } R_3 + (-1), R_2 \to R'_3 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right] \text{BY } \frac{1}{7}R3 \to R'3 \ \underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right] \begin{array}{l} By\ R_1 + (-6)R_3 \to R'_1 \\ \text{and } R_2 + 4R_3, R'_2 \end{array}$$

Thus the solution is $x = -1$, $y = 2$ and $z = 3$.
The augmented matrix for the system (2) is

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{array}\right]$$

Adding $(-2)R_1$ to R_2 and $(-3)R_1$ to R_3, we get

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{array}\right] \qquad \underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 1 & 2 & \vdots & 1 \\ 0 & -3 & 3 & \vdots & 9 \\ 0 & 2 & -2 & \vdots & -6 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 2 & -2 & \vdots & -6 \end{array}\right] \text{By } -\frac{1}{3}R_2 \to R_2' \ \underset{\sim}{R} \left[\begin{array}{ccccc} 1 & 0 & 3 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right] \begin{array}{l} \text{BY } R_1 + (-1)R_2 \to R_1' \\ \text{and } R_3 + (-2)R_2 \to R_3' \end{array}$$

The system (2) is reduced to equivalent system

$$x + 3z = 4$$
$$y - z = -3$$
$$0z = 0$$

The equation $0z = 0$ is satisfied by any value of z.

From the first and second equations, we get

$$x = -3z + 4 \qquad ....(a)$$

and $\qquad y = z - 3 \qquad ....(b)$

As z is arbitrary, so we can find infinitely many values of x and y from equation (a) and (b) or the system (2), is satisfied by $x = 4 - 3t$, $y = t - 3$ and $z = t$ for any real value of t.

Thus the system (2) has infinitely many solutions and it is consistent.

The augmented matrix of the system (3) is $\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$

Adding $(-2)R_1$, to R_2 and $(-3)R_1$, to R_3 we have

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix} \quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & -4 & 1 & \vdots & 5 \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix} \text{By } -\frac{1}{4}R_2 \to R_2' \quad \underset{\sim}{R} \begin{bmatrix} 1 & 0 & \frac{7}{4} & \vdots & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 0 & 0 & \vdots & 4 \end{bmatrix} \begin{matrix} \text{By } R_1 + 1.R_2 \to R_1' \\ \text{and } R_3 + (-8)R_2 \to R_3' \end{matrix}$$

Thus the system (3) is reduced to the equivalent system

$$x + \frac{7}{4}z = \frac{1}{4}$$
$$y - \frac{1}{4}z = \frac{5}{4}$$
$$0z = 4$$

The third equation $0z = 4$ has no solution, so the system as a whole has no solution. Thus the system is inconsistent.

We see that in the case of the system (1), the (row) rank of the augmented matrix and the coefficient matrix of the system is the same, that is, 3 which is equal to the number of the variables in the system (1).

Thus a linear system is consistent and has a unique solution if the (row) rank of the coefficient matrix is the same as that of the augmented matrix of the system.

In the case of the system (2), the (row) rank of the coefficient matrix is the same as that of the augmented matrix of the system but it is 2 which is less than the number of variables in the system (2).

Thus a system is consistent and has infinitely many solutions if the (row) ranks of the coefficient matrix and the augmented matrix of the system are equal but the rank is less than the number of variables in the system.

In the case of the system (3), we see that the (row) rank of the coefficient matrix is not equal to the (row) rank of the augmented matrix of the system.

Thus we conclude that a system is inconsistent if the (row) ranks of the coefficient matrix and the augmented matrix of the system are different.

3.11.1 Homogeneous Linear Equations

Each equation of the system of following linear equations:

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & = 0 & ....(i) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & = 0 & ....(ii) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & = 0 & ....(iii) \end{matrix}\right\}$$

is always satisfied by $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, so such a system is always consistent. The solution (0, 0, 0) of the above homogeneous equations (i), (ii), and (iii) is called the trivial solution. Any other solution of equations (i), (ii) and (iii) other than the trivial solution is called a **non-trivial**

solution. The above system can be written as

$$AX = O \text{, where } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $|A| \neq 0$, then A is non-singular and A^{-1} exists, that is,

$$A^{-1}(AX) = A^{-1}O = O$$

or $\quad (A^{-1}A)X = O \Rightarrow X = O$, i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In this case the system of homogeneous equations possesses only the trivial solution.

Now we consider the case when the system has a non-trivial solution.

Multiplying the equations (i), (ii) and (iii) by A_{11}, A_{21} and A_{31} respectively and adding the resulting equations (where A_{11}, A_{21} and A_{31} are cofactors of the corresponding elements of A), we have

$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})x_2 + (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})x_3 = 0$, that is, $|A|x_1 = 0$. Similarly, we can get $|A|x_2 = 0$ and $|A|x_3 = 0$

For a non-trivial solution, at least one of x_1, x_2 and x_3 is different from zero. Let $x_1 \neq 0$, then from $|A|x_1 = 0$, we have $|A| = 0$.

For example, the system

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \quad &(\mathrm{I}) \\ x_1 - x_2 + 3x_3 &= 0 \quad &(\mathrm{II}) \\ x_1 + 3x_2 - x_3 &= 0 \quad &(\mathrm{III}) \end{aligned}\right\}$$

has a non-trivial solution because

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0$$

Solving the first two equations of the system, we have

$2x_1 + 4x_3 = 0 \qquad$ (adding (I) and (II))

$\Rightarrow \quad x_1 = -2x_3$

and $\; 2x_2 - 2x_3 = 0 \qquad$ (subtracting (II) from (I))

$\Rightarrow \quad x_2 = x_3$

Putting $x_1 = -2x_3$ and $x_2 = x_3$ in (III), we see that $(-2x_3) + 3(x_3) - x_3 = 0$, which shows that the equation (I), (II) and (III) are satisfied by

$$x_1 = -2t,\ x_2 = t \text{ and } x_3 = t \text{ for any real value of } t.$$

Thus the system consisting of (I), (II) and (III) has infinitely many solutions. But the system

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 - 2x_3 &= 0 \end{aligned}\right\}$$ has only the trivial solution,

because in this case

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Solving the first two equations of the above system, we get $x_1 = -2x_3$ and $x_2 = x_3$. Putting $x_1 = -2x_3$ and $x_2 = x_3$ in the expression.

$x_1 + 3x_2 - 2x_3$, we have $-2x_3 + 3(x_3) - 2x_3 = -x_3$, that is,

the third equation is not satisfied by putting $x_1 = -2x_3$ and $x_2 = x_3$ but it is satisfied only if $x_3 = 0$. Thus the above system has only the trivial solution.

3.11.2 Non-Homogeneous Linear Equations

Now we will solve the systems of non-homogeneous linear equations with help of the following methods.

i) Using matrices, that is, $AX = B \Rightarrow X = A^{-1}B$.
ii) Using echelon and reduced echelon forms
iii) Using Cramer's rule.

Example 1: Use matrices to solve the system $\left.\begin{aligned} x_1 - 2x_2 + x_3 \; - &= 4 \\ 2x_1 - 3x_2 + 2x_3 - &= 6 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}\right\}$

Solution: The matrix form of the given system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$$

or $\quad AX = B \qquad$ (i)

where $\quad A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$

As $\quad |A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{vmatrix}$ By $R_2 + (-2)R_1 \to R_2'$

$$= (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1 \text{ , that is,}$$

$|A| \neq 0$, so the inverse of A exists and (i) can be written as

$X = A^{-1}B \qquad$ (ii)

Now we find adj A.

Since $\quad [A_{ij}]_{3\times 3} = \begin{bmatrix} -7 & 2 & 10 \\ 4 & -1 & -6 \\ -1 & 0 & 1 \end{bmatrix}, \left(\because A_{11} = -7, A_{12} = 2, A_{13} = 10, A_{21} = 4 \atop A_{22} = -1, A_{23} = -6, A_{31} = 1, A_{32} = 0, A_{33} = 1 \right)$

So $\quad$ adj $A = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$

and $\quad A^{-1} = \dfrac{1}{|A|}\text{adj}A = \dfrac{1}{-1}\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ 2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$

Thus $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x_1 = 1$, $x_2 = 2$ and $x_3 = -1$.

Example 2: Solve the system;

$$\left.\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3 \\ 4x_1 + 5x_2 - 3x_3 &= -3 \\ 3x_1 - 2x_2 + 17x_3 &= 42 \end{aligned}\right\},$$

by reducing its augmented matrix to the echelon form and the reduced echelon form.

Solution: The augmented matrix of the given system is

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 4 & 5 & -3 & \vdots & -3 \\ 3 & -2 & 17 & \vdots & 42 \end{bmatrix}$$

We reduce the above matrix by applying elementary row operations, that is,

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 4 & 5 & -3 & \vdots & -3 \\ 3 & -2 & 17 & \vdots & 42 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & -7 & -11 & \vdots & -15 \\ 0 & -11 & 11 & \vdots & 33 \end{bmatrix} \begin{array}{l} \text{By } R_2 + (-4)R_1 \to R_2' \\ \text{and } R_3 + (-3)R_1 \to R_3' \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & -11 & 11 & \vdots & 33 \\ 0 & -7 & -11 & \vdots & -15 \end{bmatrix} \text{By } R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & -7 & -11 & \vdots & -15 \end{bmatrix} \text{By } \left(-\frac{1}{11}\right)R_2 \to R_2' \; \underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & -18 & \vdots & -36 \end{bmatrix} \text{By } R_3 + 7R_2 \to R_3'$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \text{By } \left(-\frac{1}{18}\right)R_3 \to R_3'$$

The equivalent system in the (row) echelon form is

$$\left.\begin{array}{r} x_1 + 3x_2 + 2x_3 = 3 \\ x_2 - x_3 = -3 \\ x_3 = 2 \end{array}\right\}$$

Substituting $x_3 = 2$ in the second equation gives: $x_2 - 2 = -3 \Rightarrow x_2 = -1$
Putting $x_2 = -1$ and $x_3 = 2$ in the first equation, we have

$$x_1 + 3(-1) + 2(2) = 3 \Rightarrow x_1 = 3 + 3 - 4 = 2.$$

Thus the solution is $x_1 = 2$, $x_2 = -1$ and $x_3 = 2$

Now we reduce the matrix $\left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & 2 \end{array}\right]$ to reduced (row) echelon form, i.e.,

$$\left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 0 & 5 & 12 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{By } R_1 + (-3)R_2 \to R_1'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} \text{By } R_1 + (-5)R_3 \to R_1' \\ \text{and} R_2 + 1.R_3 \to R_2' \end{array}$$

The equivalent system in the reduced (row) echelon form is

$$x_1 = 2$$
$$x_2 = -1$$
$$x_3 = 2$$

which is the solution o f the given system.

3.12 Cramer's Rule

Consider the system of equations,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\right\} \qquad (1)$$

These are three linear equations in three variables x_1, x_2, x_3 with coefficients and constant terms in the real field R. We write the above system of equations in matrix form as: $AX = B$ (2)

where $A = [a_{ij}]_{3\times3}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

We know that
the matrix equation (2) can be written as: $X = A^{-1}B$ (if A^{-1} exists)

Note: $A^{-1}(AX) \neq BA^{-1}$

We have already proved that $A^{-1} = \dfrac{1}{|A|}\text{adj } A$ and

$$adj\ A = [A'_{ij}]_{3\times3} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \qquad (\because A'_{ij} = A_{ji})$$

Thus $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11}b_1 + A_{21}b_2 + A_{31}b_3 \\ A_{12}b_1 + A_{22}b_2 + A_{32}b_3 \\ A_{13}b_1 + A_{23}b_2 + A_{33}b_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{11}b_1 + A_{21}b_2 + A_{31}b_3}{|A|} \\ \dfrac{A_{12}b_1 + A_{22}b_2 + A_{32}b_3}{|A|} \\ \dfrac{A_{13}b_1 + A_{23}b_2 + A_{33}b_3}{|A|} \end{bmatrix}$$

Hence $$x_1 = \frac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{|A|} = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} \qquad \text{(i)}$$

$$x_2 = \frac{b_1A_{12}+b_2A_{22}+b_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} \qquad \text{(ii)}$$

$$x_3 = \frac{b_1A_{13}+b_2A_{23}+b_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|} \qquad \text{(iii)}$$

The method of solving the system with the help of results (i), (ii) and (iii) is often referred to as **Cramer's Rule.**

Example 3: Use Crammer's rule to solve the system. $\left.\begin{aligned} 3x_1+x_2-x_3&=-4 \\ x_1+x_2-2x_3&=-4 \\ -x_1+2x_2-x_3&=1 \end{aligned}\right\}$

Solution: Here $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+4)-1.(-1-2)-1.(2+1)$

$= 9 + 3 - 3 = 9$

So $x_1 = \dfrac{\begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}}{9} = \dfrac{-4(-1+4)-1(4+2)-1(-8-1)}{9}$

$$= \frac{-12-6+9}{9} = \frac{-9}{9} = 1$$

$$x_2 = \frac{\begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix}}{9} = \frac{3(4+2)+4(-1-2)-1(1-4)}{9}$$

$$= \frac{18-12+3}{9} = \frac{9}{9} = 1$$

$$x_3 = \frac{\begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix}}{9} = \frac{3(1+8)-1(1-4)-(2+1)}{9}$$

$$= \frac{27+3-12}{9} = \frac{18}{9} = 2$$

Hence $x_1 = -1$, $x_2 = 1$, $x_3 = 2$

Exercise 3.5

1. Solve the following systems of linear equations by Cramer's rule.

i) $\left.\begin{aligned} 2x+2y+z&=3 \\ 3x-2y-2z&=1 \\ 5x+y-3z&=2 \end{aligned}\right\}$ ii) $\left.\begin{aligned} 2x_1-x_2+x_3&=5 \\ 4x_1+2x_2+3x_3&=8 \\ 3x_1-4x_2-x_3&=3 \end{aligned}\right\}$ iii) $\left.\begin{aligned} 2x_1-x_2+x_3&=8 \\ x_1+2x_2+2x_3&=6 \\ x_1-2x_2-x_3&=1 \end{aligned}\right\}$

2. Use matrices to solve the following systems:

i) $\left.\begin{aligned} x-2y+z&=-1 \\ 3x+y-2z&=4 \\ y-z&=1 \end{aligned}\right\}$ ii) $\left.\begin{aligned} 2x_1+x_2+3x_3&=3 \\ x_1+x_2-2x_3&=0 \\ -3x_1-x_2+2x_3&=-4 \end{aligned}\right\}$ iii) $\left.\begin{aligned} x+y&=2 \\ 2x-z&=1 \\ 2y-3z&=-1 \end{aligned}\right\}$

3. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

i) $\left.\begin{aligned} x_1-2x_2-2x_3&=1 \\ 2x_1+3x_2+x_3&=1 \\ 5x_1-4x_2-3x_3&=1 \end{aligned}\right\}$ ii) $\left.\begin{aligned} x+2y+z&=2 \\ 2x+y+2z&=-1 \\ 2x+3y-z&=9 \end{aligned}\right\}$ iii) $\left.\begin{aligned} x_1+4x_2+2x_3&=2 \\ 2x_1+x_2-2x_3&=9 \\ 3x_1+2x_2-2x_3&=12 \end{aligned}\right\}$

4. Solve the following systems of homogeneous linear equations.

i) $\left.\begin{aligned} x+2y-2z&=0 \\ 2x+y+5z&=0 \\ 5x+4y+8z&=0 \end{aligned}\right\}$ ii) $\left.\begin{aligned} x_1+4x_2+2x_3&=0 \\ 2x_1+x_2-3x_3&=0 \\ 3x_1+2x_2-4x_3&=0 \end{aligned}\right\}$ iii) $\left.\begin{aligned} x_1-2x_2-x_3&=0 \\ x_1+x_2+5x_3&=0 \\ 2x_1-x_2+4x_3&=0 \end{aligned}\right\}$

5. Find the value of λ for which the following systems have non-trivial solutions. Also solve the system for the value of λ.

i) $\left.\begin{aligned} x+y+z&=0 \\ 2x+y-\lambda z&=0 \\ x+2y-2z&=0 \end{aligned}\right\}$ ii) $\left.\begin{aligned} x_1+4x_2+\lambda x_3&=0 \\ 2x_1+x_2-3x_3&=0 \\ 3x_1+\lambda x_2-4x_3&=0 \end{aligned}\right\}$

6. Find the value of λ for which the following system does not possessa unique solution. Also solve the system for the value of λ.

$$\left.\begin{aligned} x_1+4x_2+\lambda x_3&=2 \\ 2x_1+x_2-2x_3&=11 \\ 3x_1+2x_2-2x_3&=16 \end{aligned}\right\}$$

CHAPTER

4 Quadratic Equations

Animation 4.1: Completing the square
Source & Credit: 1ucasvb

4.1 Introduction

A *quadratic equation* in x is an equation that can be written in the form $ax^2 + bx + c = 0$; where a, b and c are real numbers and
$a \neq 0$.

Another name for a quadratic equation in x is **2nd Degree Polynomial** in x.

The following equations are the quadratic equations:

i)	$x^2 - 7x + 10 = 0;$	$a = 1,$	$b = -7,$	$c = 10$
ii)	$6x^2 + x - 15 = 0;$	$a = 6,$	$b = 1,$	$c = -15$
iii)	$4x^2 + 5x + 3 = 0;$	$a = 4,$	$b = 5,$	$c = 3$
iv)	$3x^2 - x = 0;$	$a = 3,$	$b = -1,$	$c = 0$
v)	$x^2 = 4;$	$a = 1,$	$b = 0,$	$c = -4$

4.1.1 Solution of Quadratic Equations

There are three basic techniques for solving a quadratic equation:

i) by factorization.
ii) by completing squares, extracting square roots.
iii) by applying the quadratic formula.

By Factorization: It involves factoring the polynomial $ax^2 + bx + c$.

It makes use of the fact that if $ab = 0$, then $a = 0$ or $b = 0$.

For example, if $(x - 2)(x - 4) = 0$, then either $x - 2 = 0$ or $x - 4 = 0$.

Example 1: Solve the equation $x^2 - 7x + 10 = 0$ by factorization.

Solution: $x^2 - 7x + 10 = 0$

$\Rightarrow \quad (x - 2)(x - 5) = 0$

$\therefore$ either $x - 2 = 0 \quad \Rightarrow x = 2$

or $x - 5 = 0 \quad \Rightarrow x = 5$

$\therefore$ the given equation has two solutions: 2 and 5

$\therefore$ solution set = {2, 5}

Note: The solutions of an equation are also called its roots.

$\therefore$ 2 and 5 are roots of $x^2 - 7x + 10 = 0$

By Completing Squares, then Extracting Square Roots:

Sometimes, the quadratic polynomials are not easily factorable.

For example, consider $x^2 + 4x - 437 = 0$.

It is difficult to make factors of $x^2 + 4x - 437$. In such a case the factorization and hence the solution of quadratic equation can be found by the method of completing the square and extracting square roots.

Example 2: Solve the equation $x^2 + 4x - 437 = 0$ by completing the squares.

Solution : $x^2 + 4x - 437 = 0$

$$\Rightarrow \quad x^2 + 2\left(\frac{4}{2}\right)x = 437$$

Add $\left(\frac{4}{2}\right)^2 = (2)^2$ to both sides

$x^2 + 4x + (2)^2 = 437 + (2)^2$

$\Rightarrow \quad (x + 2)^2 = 441$

$\Rightarrow \quad x + 2 = \pm\sqrt{441} = \pm 21$

$\Rightarrow \quad x = \pm 21 - 2$

$\therefore \quad x = 19$ or $x = -23$

Hence solution set = {– 23, 19}.

By Applying the Quadratic Formula:

Again there are some quadratic polynomials which are not factorable at all using integral coefficients. In such a case we can always find the solution of a quadratic equation $ax^2+bx+c = 0$ by applying a formula known as quadratic formula. This formula is applicable for every quadratic equation.

Derivation of the Quadratic Formula

Standard form of quadratic equation is

$ax^2 + bx + c = 0, a \neq 0$

Step 1. Divide the equation by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2. Take constant term to the R.H.S.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3. To complete the square on the L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence the solution of the quadratic equation $ax^2 + bx + c = 0$ is given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

which is called **Quadratic Formula.**

Example 3: Solve the equation $6x^2 + x - 15 = 0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$, we get,

$a = 6$, $b = 1$, $c = -15$

$\therefore$ The solution is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4(6)(-15)}}{2(6)}$$

$$= \frac{-1 \pm \sqrt{361}}{12} = \frac{-1 \pm 19}{12}$$

i.e., $x = \frac{-1+19}{12}$ or $x = \frac{-1-19}{12}$

$x = \frac{3}{2}$ or Hence soulation set $= \left\{\frac{3}{2}, \frac{-5}{3}\right\}$

Example 4: Solve the $8x^2 - 14x - 15 = 0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$, we get,

$a = 8$, $b = -14$, $c = -15$

By the quadratic formula, we have

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(8)(-15)}}{2(8)}$$

$$= \frac{14 \pm \sqrt{676}}{16} = \frac{14 \pm 26}{16}$$

$\therefore$ either $x = \frac{14+26}{16} \Rightarrow x = \frac{5}{2}$

or $\Rightarrow$

Hence solution set $= \left\{\frac{5}{2}, -\frac{3}{4}\right\}$

Exercise 4.1

Solve the following equations by factorization:

1. $3x^2 + 4x + 1 = 0$ **2.** $x^2 + 7x + 12 = 0$

3. $9x^2 - 12x - 5 = 0$ **4.** $x^2 - x = 2$

5. $x(x + 7) = (2x - 1)(x + 4)$

6. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{5}{2};\ x\neq -1,0$

7. $\frac{1}{x+1}+\frac{2}{x+2}=\frac{7}{x+5};\ x\neq -1,-2,-5$

8. $\frac{a}{ax-1}+\frac{b}{bx-1}=a+b;\ x\neq \frac{1}{a},\frac{1}{b}$

Solve the following equations by completing the square:

9. $x^2 - 2x - 899 = 0$ **10.** $x^2 + 4x - 1085 = 0$
11. $x^2 + 6x - 567 = 0$ **12.** $x^2 - 3x - 648 = 0$
13. $x^2 - x - 1806 = 0$ **14.** $2x^2 + 12x - 110 = 0$

Find roots of the following equations by using quadratic formula:

15. $5x^2 - 13x + 6 = 0$ **16.** $4x^2 + 7x - 1 = 0$
17. $15x^2 + 2ax - a^2 = 0$ **18.** $16x^2 + 8x + 1 = 0$
19. $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$
20. $(a + b)x^2 + (a + 2b + c)x + b + c = 0$

4.2 Solution of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic form. We shall discuss the solutions of such five types of the equations one by one.

Type I: The equations of the form: $ax^{2n} + bx^n + c = 0;\ a \neq 0$

Put $x^n = y$ and get the given equation reduced to quadratic equation in y.

Example 1: Solve the equation: $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0.$

Solution This given equation can be written as $(x^{\frac{1}{4}})^2 - x^{\frac{1}{4}} - 6 = 0$

Let $x^{\frac{1}{4}} = y$

$\therefore$ The given equation becomes

$y^2 - y - 6 = 0$
$\Rightarrow (y - 3)(y + 2) = 0$
$\Rightarrow y = 3,$ or $y = -2$
$\therefore x^{\frac{1}{4}} = 3$ $x^{\frac{1}{4}} = -2$
$\Rightarrow x = (3)^4$ $\Rightarrow x = (-2)^4$
$\Rightarrow x = 81$ $\Rightarrow x = 16$

Hence solution set is {16, 81}.

Type II: The equation of the form: $(x + a)(x + b)(x + c)(x + d) = k$
where $a + b = c + d$

Example 2: Solve $(x - 7)(x - 3)(x + 1)(x + 5) - 1680 = 0$

Solution: $(x - 7)(x - 3)(x + 1)(x + 5) - 1680 = 0$
$\Rightarrow [(x - 7)(x + 5)][(x - 3)(x + 1)] - 1680 = 0$ (by grouping)
$\Rightarrow (x^2 - 2x - 35)(x^2 - 2x - 3) - 1680 = 0$

Putting $x^2 - 2x = y$, the above equation becomes

$(y - 35)(y - 3) - 1680 = 0$
$\Rightarrow y^2 - 38y + 105 - 1680 = 0$
$\Rightarrow y^2 - 38y - 1575 = 0$

$\therefore y = \frac{38 \pm \sqrt{1444 + 6300}}{2} = \frac{38 \pm \sqrt{7744}}{2}$ (by quadratic formula)

$= \frac{38 \pm 88}{2}$

$\Rightarrow \quad y = 63 \qquad$ or $\qquad y = -25.$

$\Rightarrow \quad x^2 - 2x = 63 \qquad \Rightarrow x^2 - 2x = -25$

$\Rightarrow \quad x^2 - 2x - 63 = 0 \qquad \Rightarrow x^2 - 2x + 25 = 0$

$\Rightarrow \quad (x+7)(x-9) = 0 \qquad \Rightarrow x = \dfrac{2 \pm \sqrt{4-100}}{2}$

$\Rightarrow \quad x = -7$ or $x = 9$

$$= \frac{2 \pm \sqrt{-96}}{2}$$

$$= \frac{2 \pm 4\sqrt{6}\,i}{2} = 1 \pm 2\sqrt{6}\,i$$

$\Rightarrow$ or

Hence Solution set = $\{-7, 9, 1+2\sqrt{6}\,i, 1-2\sqrt{6}\,i\}$

Type III: Exponential Equations: Equations, in which the variable occurs in exponent, are called **exponential equations**. The method of solving such equations is explained by the following examples.

Example 3: Solve the equation: $2^{2x} - 3.2^{x+2} + 32 = 0$

Solution: $2^{2x} - 3.2^{x+2} + 32 = 0$

$\Rightarrow \quad 2^{2x} - 3.2^2 . 2^x + 32 = 0$

$\Rightarrow \quad 2^{2x} - 12.2^x + 32 = 0$

$\Rightarrow \quad y^2 - 12y + 32 = 0 \qquad$ (Putting $2^x = y$)

$\Rightarrow \quad (y-8)(y-4) = 0$

$\Rightarrow \quad y = 8 \qquad$ or $\qquad y = 4$

$\Rightarrow \quad 2^x = 8 \qquad \Rightarrow 2^x = 4$

$\Rightarrow \quad 2^x = 2^3 \qquad \Rightarrow 2^x = 2^2$

$\Rightarrow \quad x = 3 \qquad \Rightarrow x = 2$

Hence solution set = {2, 3}.

Example 4: Solve the equation: $4^{1+x} + 4^{1-x} = 10$

Solution: Given that

$4^{1+x} + 4^{1-x} = 10$

$\Rightarrow \quad 4.4^x + 4.4^{-x} = 10$

Let $4^x = y \Rightarrow 4^{-x} = (4^x)^{-1} = y^{-1} = \dfrac{1}{y}$

$\therefore$ The given equation becomes

$$4y + \frac{4}{y} - 10 = 0$$

$\Rightarrow \quad 4y^2 - 10y + 4 = 0$

$\Rightarrow \quad 2y^2 - 5y + 2 = 0$

$$\therefore \quad y = \frac{5 \pm \sqrt{25 - 4(2)(2)}}{2(2)} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$\Rightarrow \quad y = 2 \qquad$ or $\qquad y = \dfrac{1}{2}$

$\therefore \quad 4^x = 2 \qquad \therefore \quad 4^x = \dfrac{1}{2}$

$\Rightarrow \quad 2^{2x} = 2^1$

$\Rightarrow \quad 2x = 1 \qquad \Rightarrow \quad 2^{2x} = 2^{-1}$

$\Rightarrow \quad 2x = -1$

$\Rightarrow \quad x = \dfrac{1}{2}$

$\Rightarrow$

Hence Solution set = $\left\{\dfrac{1}{2}, -\dfrac{1}{2}\right\}$.

Type IV: Reciprocal Equations: An equation, which remains unchanged when x is replaced by is called a reciprocal equation. In such an equation the coefficients of the terms equidistant from the beginning and end are equal in magnitude. The method of solving such equations is explained through the following example:

Example 5: Solve the equation
$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$;

Solution: Given that:
$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

$\Rightarrow \quad x^2 - 3x + 4 - \frac{3}{x} + \frac{1}{x^2} = 0$ (Dividing by x^2)

$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 4 = 0$ (1)

Let $\Rightarrow$

So, the equation (1) reduces to

$y^2 - 2 - 3y + 4 = 0$

$\Rightarrow \quad y^2 - 3y + 2 = 0$

$\Rightarrow \quad (y - 2)(y - 1) = 0$

$\Rightarrow \quad y = 2$ or $y = 1$

$\Rightarrow \quad x + \frac{1}{x} = 2 \quad \Rightarrow \quad x + \frac{1}{x} = 1$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \quad \Rightarrow \quad x^2 - x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0 \quad \Rightarrow$

$\Rightarrow \quad (x - 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, 1 \quad \Rightarrow$

Hence Solution set

Exercise 4.2

Solve the following equations:

1. $x^4 - 6x^2 + 8 = 0$ **2.** $x^{-2} - 10 = 3x^{-1}$

3. $x^6 - 9x^3 + 8 = 0$ **4.** $8x^6 - 19x^3 - 27 = 0$

5. $x^{\frac{2}{5}} + 8 = 6x^{\frac{1}{5}}$ **6.** $(x+1)(x+2)(x+3)(x+4) = 24$

7. $(x-1)(x+5)(x+8)(x+2) - 880 = 0$

8. $(x-5)(x-7)(x+6)(x+4) - 504 = 0$

9. $(x-1)(x-2)(x-8)(x+5) + 360 = 0$

10. $(x+1)(2x+3)(2x+5)(x+3) = 945$

Hint: $(x+1)(2x+5)(2x+3)(x+3) = 945$

11. $(2x-7)(x^2-9)(2x+5) - 91 = 0$

12. $(x^2+6x+8)(x^2+14x+48) = 105$

13. $(x^2+6x-27)(x^2-2x-35) = 385$

14. $4 \cdot 2^{2x+1} - 9.2^x + 1 = 0$

15. $2^x + 2^{-x+6} - 20 = 0$

16. $4^x - 3.2^{x+3} + 128 = 0$

17. $3^{2x-1} - 12.3^x + 81 = 0$ **18.**

19. $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$ **20.** $\left(x - \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 0$

21. $2x^4 - 3x^3 - x^2 - 3x + 2 = 0$ **22.** $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

23. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ **24.** $x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$

Type V: Radical Equations: Equations involving **radical expressions** of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical-free equation but the new equation have solutions that are not solutions of the original radical equation.

Such extra solutions (roots) are called **extraneous roots**. The method of the solution of different types of radical equations is illustrated by means of the followings examples:

i) The Equations of the form: $l(ax^2+bx)+m\sqrt{ax^2+bx+c}=0$

Example 1: Solve the equation

$$3x^2+15x-2\sqrt{x^2+5x+1}=2$$

Solution : Let $\sqrt{x^2+5x+1}=y$

$\Rightarrow \quad x^2+5x+1=y^2$

$\Rightarrow \quad x^2+5x=y^2-1$

$\Rightarrow \quad 3x^2+15x=3y^2-3$

$\therefore$ The given equation becomes $3y^2-3-2y=2$

$\Rightarrow \quad 3y^2-2y-5=0$

$\Rightarrow \quad (3y-5)(y+1)=0$

$\Rightarrow \quad y=\frac{5}{3}$ or $y=-1$

$\Rightarrow \quad \sqrt{x^2+5x+1}=\frac{5}{3}$ $\qquad \Rightarrow \quad \sqrt{x^2+5x+1}=-1$

$\Rightarrow \quad x^2+5x+1=\frac{25}{9}$ $\qquad \Rightarrow \quad x^2+5x+1=1$

$\Rightarrow \quad 9x^2+45x+9=25$ $\qquad \Rightarrow \quad x^2+5x=0$

$\Rightarrow \quad 9x^2+45x-16=0$ $\qquad \Rightarrow \quad x(x+5)=0$

$\Rightarrow \quad (3x+16)(3x-1)=0$ $\qquad \therefore \quad x=0$ or $x=-5$

$\therefore \quad x=\frac{1}{3}$ or $x=-\frac{16}{3}$

On checking, it is found that 0 and – 5 do not satisfy the given equation. Therefore 0 and –5 being extraneous roots cannot be included in solution set.

Hence solution set

ii) The Equation of the form : $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Example 2: Solve the equation:

Solution: $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$

Squaring both sides, we get

$$x+8+x+3+2\sqrt{x+8}\sqrt{x+3}=12x+13$$

$\Rightarrow \quad 2\sqrt{x+8}\sqrt{x+3}=10x+2$

$\Rightarrow \quad \sqrt{(x+8)(x+3)}=5x+1$

Squaring again, we have

$x^2+11x+24=25x^2+10x+1$

$\Rightarrow \quad 24x^2-x-23=0$

$\Rightarrow \quad (24x+23)(x-1)=0$

$\Rightarrow \quad x=-\frac{23}{24}$ or $x=1$

On checking we find that $-\frac{23}{24}$ is an extraneous root. Hence solution set = {1}.

iii) The Equations of the form:

$$\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=\sqrt{lx^2+mx+n}$$

where ax^2+bx+c, px^2+qx+r and lx^2+mx+n have a common factor.

Example3: Solve the equation: $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$

Solution: Consider that:

$x^2+4x-21=(x+7)(x-3)$

$x^2-x-6=(x+2)(x-3)$

$6x^2-5x-39=(6x+13)(x-3)$

$\therefore$ The given equation can be written as

$$\sqrt{(x+7)(x-3)}+\sqrt{(x+2)(x-3)}=\sqrt{(6x+13)(x-3)}$$

$\Rightarrow \quad \sqrt{x-3}\left[\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}\right]=0$

$\therefore$ Either $\sqrt{x-3}=0$ or $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\sqrt{x-3}=0 \Rightarrow x-3 = 0 \Rightarrow x = 3$

Now solve the equation $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\Rightarrow \quad \sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

$\Rightarrow \quad x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$ (Squaring both sides)

$\Rightarrow \quad 2\sqrt{(x+7)(x+2)}=4x+4$

$\Rightarrow \quad \sqrt{x^2+9x+14}=2x+2$

$\Rightarrow \quad x^2+9x+14 = 4x^2+8x+4$ (Squaring both sides again)

$\Rightarrow \quad 3x^2-x-10=0$

$\Rightarrow \quad (3x+5)(x-2)=0$

$\Rightarrow \quad x=-\frac{5}{3},2$

Thus possible roots are 3, 2, $-\frac{5}{3}$.

On verification, it is found that $-\frac{5}{3}$ is an extraneous root. Hence solution set = {2, 3}

iv) The Equations of the form: $\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=mx+n$
where, $(mx+n)$ is a factor of $(ax^2+bx+c)-(px^2+qx+r)$

Example 4: Solve the equation: $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$

Solution: Let $\sqrt{3x^2-7x-30}=a$ and $\sqrt{2x^2-7x-5}=b$

Now $a^2-b^2=(3x^2-7x-30)-(2x^2-7x-5)$

$a^2-b^2=x^2-25 \quad$ (i)

The given equation can be written as:

$a-b=x-5 \quad$ (ii)

$\frac{(a+b)(a-b)}{a-b}=\frac{(x+5)(x-5)}{x-5}$ [From (i) and (ii)]

$\Rightarrow \quad a+b=x+5 \quad$ (iii)

$2a=2x$ [From (ii) and (iii)]

$\Rightarrow \quad a=x$

$\therefore \quad \sqrt{3x^2-7x-30}=x$

$\Rightarrow \quad 3x^2-7x-30=x^2$

$\Rightarrow \quad 2x^2-7x-30=0$

$\Rightarrow \quad (2x+5)(x-6)=0$

$\Rightarrow \quad x=-\frac{5}{2},6$

On checking, we find that $-\frac{5}{2}$ is an extraneous root.
Hence solution set = { 6 }

Exercise 4.3

Solve the following equations:

1. $3x^2+2x-\sqrt{3x^2+2x-1}=3$

2. $x^2 - - - 7 = x-3\sqrt{2x^2-3x+2}$

3. $\sqrt{2x+8}+\sqrt{x+5}=7$

4. $\sqrt{3x+4}=2+\sqrt{2x-4}$

5. $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

6. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

7. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

8. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$

9. $\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$

10. $(x+4)(x+1)=\sqrt{x^2+2x-15}+3x+31$

11. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13$

12. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

4.3 Three Cube Roots of Unity

Let x be a cube root of unity

$\therefore \quad x=\sqrt[3]{1}=(1)^{\frac{1}{3}}$

$\Rightarrow \quad x^3=1$

$\Rightarrow \quad x^3-1=0$

$\Rightarrow \quad (x-1)(x^2+x+1)=0$

Either $x-1=0 \Rightarrow x=1$

or $x^2+x+1=0$

$\therefore \quad x=\dfrac{-1\pm\sqrt{1-4}}{2} \quad =\dfrac{-1\pm\sqrt{-3}}{2}$

$\Rightarrow \quad x=\dfrac{-1\pm\sqrt{3}i}{2} (\because \sqrt{-1}=i)$

Thus the three cube roots of unity are:

$1, \dfrac{-1+\sqrt{3}i}{2}$ and $\dfrac{-1-\sqrt{3}i}{2}$

Note: We know that the numbers containing i are called **complex** numbers. So $\dfrac{-1+\sqrt{3}i}{2}$ and $\dfrac{-1-\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

*By complex root we mean, a root containing non-zero imaginary part.

4.3.1 Properties of Cube Roots of Unity

i) Each complex cube root of unity is square of the other

Proof: (a) $\left(\dfrac{-1+\sqrt{3}\,i}{2}\right)^2=\dfrac{(-1)^2+(\sqrt{3}\,i)^2+2(-1)(\sqrt{3}\,i)}{4}$

$==\dfrac{1-3-2\sqrt{3}\,i}{4} \quad \dfrac{-2-2\sqrt{3}\,i}{4}$

$=2\left(\dfrac{-1-\sqrt{3}\,i}{4}\right)$

$=\dfrac{-1-\sqrt{3}\,i}{2}$

(b) $\left(\dfrac{-1-\sqrt{3}i}{2}\right)^2=\left[-\dfrac{(1+\sqrt{3}i)}{2}\right]^2$

$=\dfrac{(1)^2+(\sqrt{3}\,i)^2+(2)(1)(\sqrt{3}\,i)}{4}$

$==\dfrac{1-3+2\sqrt{3}i}{4} \quad \dfrac{-2+2\sqrt{3}i}{4}$

$=2\left(\dfrac{-1+\sqrt{3}\,i}{4}\right)$

$=\dfrac{-1+\sqrt{3}\,i}{2}$

Hence each complex cube root of unity is square of the other.

Note: if $\dfrac{-1+\sqrt{3}\,i}{2}=\omega$, then $\dfrac{-1-\sqrt{3}\,i}{2}=\omega^2$,

and if $\dfrac{-1-\sqrt{3}\,i}{2}=\omega$, then $\dfrac{-1+\sqrt{3}\,i}{2}=\omega^2$ [ω is read as omega]

ii) The Sum of all the three cube roots of unity is zeroi.e. $1+\omega+\omega^2=0$

Proof: We know that cube roots of unity are

$1, \dfrac{-1+\sqrt{3}\,i}{2}$ and $\dfrac{-1-\sqrt{3}\,i}{2}$

Sum of all the three cube roots = $1+\dfrac{-1+\sqrt{3}\,i}{2}+\dfrac{-1-\sqrt{3}\,i}{2}$

$=\dfrac{2-1+\sqrt{3}\,i-1-\sqrt{3}\,i}{2}=\dfrac{0}{2}=0$

if $\omega==\dfrac{-1+\sqrt{3}i}{2}$, then $\omega^2 \quad \dfrac{-1-\sqrt{3}i}{2}$

Hence sum of cube roots of unity $=1+\omega+\omega^2=0$

iii) The product of all the three cube roots of unity is unity i.e., $\omega^3 = 1$

Proof: Let $\dfrac{-1+\sqrt{3}i}{2} = \omega$ and $\dfrac{-1-\sqrt{3}i}{2}$ ω^2

$$\therefore \quad 1.\omega.\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= \frac{(-1)^2 - (\sqrt{3}i)^2}{4}$$

$$= \frac{1-(-3)}{4} = \frac{1+3}{4}$$

$$\Rightarrow \quad \omega^3 = 1$$

$\therefore$ Product of the complex cube roots of unity $= \omega^3 = 1$.

iv) For any $n \in z, \omega^n$ is equivalent to one of the cube roots of unity.

With the help of the fact that $\omega^3 = 1$, we can easily reduce the higher exponent of ω to its lower equivalent exponent.

$$\begin{aligned}
\text{e.g.} \quad \omega^4 &= \omega^3.\omega = 1.\omega = \omega \\
\omega^5 &= \omega^3.\omega^2 = 1.\omega^2 = \omega^2 \\
\omega^6 &= (\omega^3)^2 = (1)^2 = 1 \\
\omega^{15} &= (\omega^3)^5 = (1)^5 = 1 \\
\omega^{27} &= (\omega^3)^9 = (1)^9 = 1 \\
\omega^{11} &= \omega^9.\omega^2 = (\omega^3)^3.\ \omega^2 = (1)^3.\omega^2 = \omega^2 \\
\omega^{-1} &= \omega^{-3}.\omega^2 = (\omega^3)^{-1}.\omega^2 = \omega^2 \\
\omega^{-5} &= \omega^{-6}.\omega = (\omega^3)^{-2}.\omega = \omega \\
\omega^{-12} &= (\omega^3)^{-4} = (1)^{-4} = 1
\end{aligned}$$

Example 1: Prove that: $(x^3+y^3) = (x+y)(x+\omega y)(x+\omega^2 y)$

Solution :

$$\begin{aligned}
\text{R.H.S} &= (x+y)(x+\omega y)(x+\omega^2 y) \\
&= (x+y)[x^2 + (\omega+\omega^2)yx + \omega^3 y^2] \\
&= (x+y)(x^2 - xy + y^2) = x^3 + y^3 \qquad \{\because \omega^3 = 1, \omega+\omega^2 = -1\} \\
&= \text{L.H.S.}
\end{aligned}$$

Example 2: Prove that: $= (-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 = -16$

Solution:

$$\begin{aligned}
\text{L.H.S} &= (-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 \\
&= \left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^4 \quad \left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^4 \\
&= (2\omega)^4 + (2\omega^2)^4 \qquad \left\{ \text{Let } \frac{-1+\sqrt{-3}}{2} = \omega \quad \therefore \frac{-1-\sqrt{-3}}{2} = \omega^2 \right. \\
&= 16\omega^4 + 16\omega^8 \\
&= 16(\omega^4 + \omega^8) \\
&= 16\left[\omega^3.\ \omega + \omega^6.\ \omega^2\right] \\
&= 16(\omega+\omega^2) \qquad \because \omega^3 = \omega^6 = 1 \\
&= 16(-1) \qquad \because \omega + \omega^2 = \quad 1 \\
&= -16 = \text{R.H.S}
\end{aligned}$$

4.4 Four Fourth Roots of Unity

Let x be the fourth root of unity

$$\therefore \quad x = \sqrt[4]{1} \quad (1)^{\frac{1}{4}}$$

$$\Rightarrow \quad x^4 = 1$$

$$\Rightarrow \quad x^4 - 1 = 0$$

$$\Rightarrow \quad (x^2-1)(x^2+1) = 0$$

$$\Rightarrow \quad x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\text{and } x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i.$$

Hence four fourth roots of unity are:

+ 1 ,– 1 , + i, – i.

4.4.1 Properties of four Fourth Roots of Unity

We have found that the four fourth roots of unity are:

$+1, -1, +i, -i$

i) Sum of all the four fourth roots of unity is zero

$\because \quad +1+(-1)+i+(-i)=0$

ii) The real fourth roots of unity are additive inverses of each other

+1 and –1 are the real fourth roots of unity

and +1 + (–1) = 0 = (–1) + 1

iii) Both the complex/imaginary fourth roots of unity are conjugate of each other

i and $-i$ are complex / imaginary fourth roots of unity, which are obviously conjugates of each other.

iv) Product of all the fourth roots of unity is –1

$\therefore \quad 1\times(-1)\times i\times(-i)=-1$

Exercise 4.4

1. Find the three cube roots of: 8, – 8, 27, –27, 64.

2. Evaluate:

i) $(1+\omega-\omega^2)^8$ ii) $\omega^{28}+\omega^{29}+1$ iii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

iv) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^7$ v) $(-1+\sqrt{-3})^5+(-1-\sqrt{3})^5$

3. Show that:

i) $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

ii) $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

iii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots.2n \text{ factors}=1$

Hint: $1+\omega^4=1+\omega^3.\omega=1+\omega=-\omega^2,\ 1+\omega^8=1+\omega^6.\omega^2=1+\omega^2=-\omega$

4. If ω is a root of $x^2+x+1=0$, show that its other root is ω^2 and prove that $\omega^3=1$.

5. Prove that complex cube roots of –1 are $\dfrac{1+\sqrt{3}i}{2}$ and $\dfrac{1-\sqrt{3}i}{2}$ and hence prove that

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9 = 2..$$

6. If ω is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$.

7. Find four fourth roots of 16, 81, 625.

8. Solve the following equations:

i) $2x^4-32=0$ ii) $3y^5-243y=0$

iii) $x^3+x^2+x+1=0$ iv) $5x^5-5x=0$

4.5 Polynomial Function:

A polynomial in x is an expression of the form

$$a_nx^n+a_{n-1}x^{n-1}+\ldots.+a_{1n}x+a_0, \qquad a_n\neq 0 \qquad \text{(i)}$$

where n is a non-negative integer and the coefficients $a_n, a_{n-1}, \ldots., a_1$ and a_0 are real numbers. It can be considered as a **Polynomial function** of x. The highest power of x in polynomial in x are called the **degree** of the polynomial. So the expression (i), is a polynomial of degree n. The polynomials $x^2-2x+3,\ 3x^3+2x^2-5x+4$ are of degree 2 and 3 respectively.

Consider a polynomial; $3x^3-10x^2+13x-6.$

If we divide it by a linear factor $x-2$ as shown below, we get a quotient x^2-4x+5 and a remainder 4 .

$$
\begin{array}{rl}
 & \quad 3x^2 - 4x + 5 \\
\text{divisor} \rightarrow x-2 \big) & 3x^3 - 10x^2 + 13x - 6 \quad \leftarrow \text{dividend} \\
 & 3x^3 - 6x^2 \\
 & - \quad + \\
 \hline
 & \quad -4x^2 + 13x \\
 & \quad -4x^2 + 8x \\
 & \quad + \quad - \\
 \hline
 & \qquad 5x - 6 \\
 & \qquad 5x - 10 \\
 & \qquad - \quad + \\
 \hline
 & \qquad\quad 4 \quad \leftarrow \text{remainder}
\end{array}
$$

Hence we can write: $3x^3 - 10x^2 + 13x - 6 = (x-2)(3x^2 - 4x + 5) + 4$

i.e., dividend = (divisor) (quotient) + remainder.

4.6 Theorems:

Remainder Theorem: If a polynomial $f(x)$ of degree $n \geq 1$, n is non-negative integer is divided by $x - a$ till no x-term exists in the remainder, then $f(a)$ is the remainder.

Proof: Suppose we divide a polynomial $f(x)$ by $x - a$. Then there exists a unique quotient $q(x)$ and a unique remainder R such that $f(x) = (x - a)(qx) + R$ (i)

Substituting $x = a$ in equation (i), we get

$f(a) = (a - a)q(a) + R$

$\Rightarrow \quad f(a) = R$

Hence remainder = $f(a)$

Note: Remainder obtained when $f(x)$ is divided by $x - a$ is same as the value of the polynomial $f(x)$ at $x = a$.

Example 1: Find the remainder when the polynomial $x^3 + 4x^2 - 2x + 5$ is divided by $x - 1$.

Solution: Let $f(x) = x^3 + 4x^2 - 2x + 5$ and $x - a = x - 1 \Rightarrow a = 1$

Remainder = $f(1)$ (By remainder theorem)

$= (1)^3 + 4(1)^2 - 2(1) + 5$

$= 1 + 4 - 2 + 5$

$= 8$

Example 2: Find the numerical value of k if the polynomial $x^3 + kx^2 - 7x + 6$ has a remainder of -4, when divided by $x + 2$.

Solution: Let $f(x) = x^3 + kx^2 - 7x + 6$ and $x - a = x + 2$, we have, $a = -2$

Remainder = $f(-2)$ (By remainder theorem)

$= (-2)^3 + k(-2)^2 - 7(-2) + 6$

$= -8 + 4k + 14 + 6$

$= 4k + 12$

Given that remainder = -4

$\therefore \quad 4k + 12 = -4$

$\Rightarrow \quad 4k = -16$

$\Rightarrow \quad k = -4$

Factor Theorem: The polynomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$ i.e.; $(x - a)$ is a factor of $f(x)$ if and only if $x = a$ is a root of the polynomial equation $f(x) = 0$.

Proof: Suppose $g(x)$ is the quotient and R is the remainder when a polynomial $f(x)$ is divided by $x - a$, then by **Remainder Theorem**

$f(x) = (x-a)\ g(x) + R$

Since $f(a) = 0 \quad \Rightarrow R = 0$

$\therefore \quad f(x) = (x-a)\ g(x)$

$\therefore \quad (x-a)$ is a factor of $f(x)$.

Conversely, if $(x-a)$ is a factor of $f(x)$, then

$R = f(a) = 0$

which proves the theorem.

Note: To determine if a given linear polynomial $x - a$ is a factor of $f(x)$, all we need to check whether $f(a) = 0$.

Example 3: Show that $(x - 2)$ is a factor of $x^4 - 13x^2 + 36$.

Solution: Let $f(x) = x^4 - 13x^2 + 36$ and $x - a = x - 2 \Rightarrow a = 2$

Now $f(2) = (2)^4 - 13(2)^2 + 36$

$= 16 - 52 + 36$

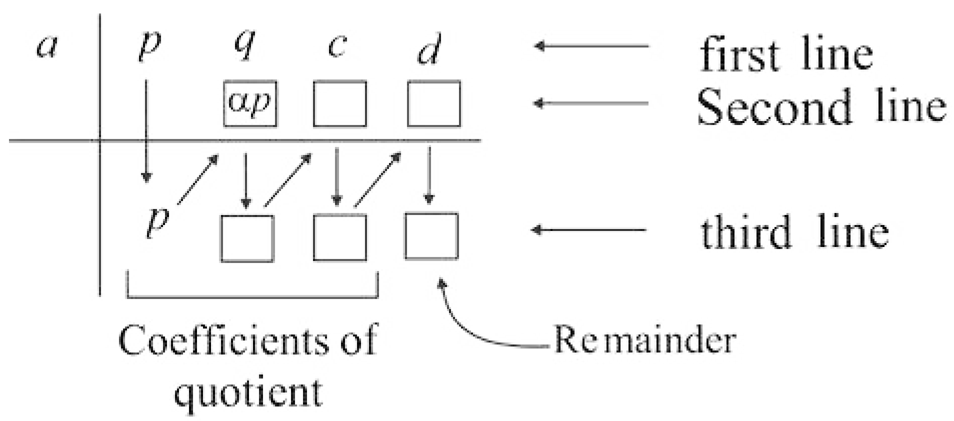
$= 0 =$ remainder

$\Rightarrow (x - 2)$ is a factor of $x^4 - 13x^2 + 36$

4.7 Synthetic Division

There is a nice shortcut method for long division of a polynomial $f(x)$ by a polynomial of the form $x - a$. This process of division is called **Synthetic Division**.

To divide the polynomial $px^3 + qx^2 + cx + d$ by $x - a$



Out Line of the Method:

i) Write down the coefficients of the dividend $f(x)$ from left to right in decreasing order of powers of x. Insert 0 for any missing terms.

ii) To the left of the first line, write a of the divisor $(x - a)$.

iii) Use the following patterns to write the second and third lines:

Vertical pattern $(\downarrow)$ Add terms

Diagonal pattern $(\nearrow)$ Multiply by a.

Example 4: Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^2 - 2x + 4$ is divided by $x + 3$.

Solution: Let $f(x) = x^4 - 10x^2 - 2x + 4$

$= x^4 + 0x^3 - 10x^2 - 2x + 4$

and $x - a = x + 3 = x - (-3) \Rightarrow x = -3$

Dividend $x^4 - 10x^2 - 2x + 4$

-3	1	0	-10	-2	4	first line
		-3	9	3	-3	Second line
	1	-3	-1	1	$\boxed{1}$	third line
					Remainder	

$\therefore$ Quotient $= x^3 - 3x^2 - x + 1$

Remainder $= 1$

Example 5: If $(x - 2)$ and $(x + 2)$ are factors of $x^4 - 13x^2 + 36$. Using synthetic division, find the other two factors.

Solution: Let $f(x) = x^4 - 13x^2 + 36$

$= x^4 + 0x^3 - 13x^2 - 0x + 36$

Here $x - a = x - 2 \Rightarrow x = 2$ and $x - a = x + 2 = x - (-2) \Rightarrow x = -2$

By synthetic Division:

2	1	0	−13	0	36	
		2	4	−18	−36	
−2	1	2	−9	−18	0	Remainder
		−2	0	18		
	1	0	−9	0		Remainder

$\therefore$ Quotient $= x^2 + 0x - 9$

$= x^2 - 9$

$= (x+3)(x-3)$

$\therefore$ Other two factors are $(x+3)$ and $(x-3)$.

Example 6: If $x+1$ and $x-2$ are factors of x^3+px^2+qx+2. By use of synthetic division find the values of p and q.

Solution: Here $x-a = x+1 \Rightarrow a=-1$ and $x-a = x-2 \Rightarrow a = 2$

Let $f(x) = x^3+px^2+qx+2$

By Synthetic Division:

-1	1	p	q	2
		-1	$-p+1$	$-q+p-1$
2	1	$p-1$	$q-p+1$	$1-q+p$
		2	$2p+2$	
	1	$p+1$	$p+q+3$	

Remainder

Since $x+1$ and $x-2$ are the factors of $f(x)$

$\therefore$ $p-q+1=0$ (i)

and $p+q+3=0$ (ii)

Adding (i) & (ii) we get $2p+4=0$ $\Rightarrow p=-2$

from (i) $-2-q+1=0$ $\Rightarrow q=-1$

Example 7: By the use of synthetic division, solve the equation $x^4-5x^2+4=0$ if -1 and 2 are its roots.

-1	1	0	-5	0	4
		-1	1	4	-4
2	1	-1	-4	4	0
		2	2	-4	
	1	1	-2	0	

Remainder

Solution: $f(x) = x^4-0x^3-5x^2+0x+4$

Depressed Equation:

$x^2+x-2=0$

$\Rightarrow (x+2)(x-1)=0$ $\Rightarrow x=-2$ or $x=1$

Hence Solution set = $\{-2, -1, 1, 2\}$.

Exercise 4.5

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial:

1. x^2+3x+7, $x+1$

2. x^3-x^2+5x+4, $x-2$

3. $3x^4+4x^3+x-5$, $x+1$

4. x^3-2x^2+3x+3, $x-3$

Use the factor theorem to determine if the first polynomial is *a* factor of the second polynomial.

5. $x-1$, x^2+4x-5

6. $x-2$, x^3+x^2-7x+1

7. $\omega+2$, $2\omega^3+\omega^2-4\omega+7$

8. $x-a$, x^n-a^n where n is a positive integer

9. $x+a$, x^n+a^n where n is an odd integer.

10. When $x^4+2x^3+kx^2+3$ is divided by $x-2$ the remainder is 1. Find the value of k.

11. When the polynomial x^3+2x^2+kx+4 is divided by $x-2$ the remainderis 14. Find the value of k.

Use Synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

12. $x^3-7x+6=0,\quad x=2$

13. $x^3-28x-48=0, - x = 4$

14. $2x^4+7x^3-4x^2-27x-18$, $x=2, -\ x=3$

15. Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of the polynomial x^3+px^2+qx+6.

16. Find the values of a and b if –2 and 2 are the roots of the polynomial x^3-4x^2+ax+b.

4.8 Relations Between the Roots and the Coefficients of a Quadratic Equation

Let α,β are the roots of $ax^2+bx+c=0$, $a\neq 0$ such that

$$\frac{-b+\sqrt{b\ \ -\ \ ac}}{} \quad \text{and} \quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ \alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a}\quad\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}\quad -\quad -\frac{2b}{2a}\quad\frac{b}{a}$$

$$\text{and}\quad \alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}\quad\frac{4ac}{4a^2}\quad\frac{c}{a}$$

Sum of the roots = S $-\frac{b}{a}=-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of the roots = $P=\frac{c}{a}=\frac{\text{constant term}}{\text{coefficient of } x^2}$

The above results are helpful in expressing symmetric functions of the roots in terms of the coefficients of the quadratic equations.

Example 1: If α,β are the roots of $ax^2+bx+c=0$, $a\neq 0$, find the values of

i) $\alpha^2+\beta^2$ ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ iii) $(\alpha-\beta)^2$

Solution: Since α,β are the roots of $ax^2+bx+c=0$

$$\therefore\ \alpha+\beta=-\frac{b}{a}\quad\text{and}\quad\alpha\beta=\frac{c}{a}$$

i) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ca}{a^2}$$

ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$=\frac{\left(-\frac{b}{a}\right)^3-3\frac{c}{a}\left(-\frac{b}{c}\right)}{\frac{c}{a}}\quad\frac{\frac{-b^3+3abc}{a^3}}{\frac{c}{a}}$$

$$=\frac{-b^3+3abc}{a^2c}$$

iii) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-4\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-4\frac{c}{a}=\frac{b^2-4ac}{a^2}$$

Example 2: Find the condition that one root of $ax^2+bx+c=0$, $a\neq 0$ is square of the other.

Solution: As one root of $ax^2+bx+c=0$ is square of the other, let the roots be α and α^2

Sum of roots $a+a^2=-\frac{b}{a}$ (i)

Product of roots= $a\,.\,a^2=\frac{c}{a}$ $\Rightarrow\alpha^3=\frac{c}{a}$ (ii)

Cubing both sides of (i), we get

$$a^3+a^6+3aa^2(a+a^2)=-\frac{b^3}{a^3}$$

$$\Rightarrow \quad a^3+(a^3)^2+3a^3(a+a^2)= \frac{b^3}{a^3}$$

$$\Rightarrow \quad \frac{c}{a}+\left(\frac{c}{a}\right)^2+3\frac{c}{a}\left(-\frac{b}{a}\right)= \frac{b^3}{a^3} \qquad \text{(From (i), (ii))}$$

$$\Rightarrow \quad a^2c+ac^2-3abc= b^3$$

4.9 Formation of an Equation Whose Roots are Given

$\therefore \quad (x-a)(x-\beta)=0$ has the roots α and β

$\Rightarrow \quad x^2-(a+\beta)x+a\beta=0$ has the roots α and β.

For S = Sum of the roots and P = Product of the roots.

Thus $\boxed{x^2 - Sx + P = 0}$

Example 3: If α , β are the root of $ax^2+bx+c=0$ form the equation whose roots are double the roots of this equation.

Solution: $\because$ α and β are the root of $ax^2+bx+c=0$

$$\therefore \quad \alpha+\beta - = \frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

The new roots are 2α and 2β.

$\therefore$ Sum of new roots = $2\alpha + 2\beta$

$$= 2(\alpha+\beta) = -\frac{2b}{a}$$

Product of new roots= $2\alpha . 2\beta = 4\alpha\beta = \frac{4c}{a}$

Required equation is given by

y^2 –(Sum of roots) y + Product of roots = 0

$$\Rightarrow \quad y^2+\frac{2b}{a}y+\frac{4c}{a}=0 \qquad \Rightarrow \quad ay^2+2by+4c=0$$

Exercise 4.6

1. If α , β are the root of $3x^2 - 2x + 4 = 0$, find the values of

i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ **ii)** $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ **iii)** $a^4+\beta^4$

iv) $a^3+\beta^3$ **v)** $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ **vi)** $a^2-\beta^2$

2. If α , β are the root of $x^2 - px - p - c = 0$, prove that
$(1+\alpha)(1+\beta) = 1 - c$

3. Find the condition that one root of $x^2+px+q=0$ is

i) double the other **ii)** square of the other

iii) additive inverse of the other

iv) multiplicative inverse of the other.

4. If the roots of the equation $x^2-px+q=0$ differ by unity, prove that $p^2 = 4q + 1$.

5. Find the condition that $\frac{a}{x-a}+\frac{b}{x-b}=5$ may have roots equal in magnitude but opposite in signs.

6. If the roots of $px^2+qx+q=0$ are α and β then prove that $\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}}=0$.

7. If α , β are the roots of the equation $ax^2+bx+c=0$, form the equations whose roots are

i) a^2,β^2 **ii)** $\frac{1}{\alpha},\frac{1}{\beta}$ **iii)** $\frac{1}{\alpha^2},\frac{1}{\beta^2}$

iv) a^3,β^3 **v)** $\frac{1}{\alpha^3},\frac{1}{\beta^3}$ **vi)** $\alpha+\frac{1}{\alpha},\beta+\frac{1}{\beta}$

vii) $(\alpha-\beta)^2, (\alpha+\beta)^2$ viii) $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

8. If α , β are the roots of the $5x^2 - x - 2 = 0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$.

9. If α , β are the roots of the $x^2 - 3x + 5 = 0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$.

4.10 Nature of the roots of a quadratic equation

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula as: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

We see that there are two possible values for x, as discriminated by the part of the formula $\pm\sqrt{b^2-4ac}$.

The nature of the roots of an equation depends on the value of the expression $b^2 - 4ac$, which is called its **Discriminant.**

Case 1: If $b^2 - 4ac = 0$ then the roots will be $-\frac{b}{2a}$ and $-\frac{b}{2a}$ So, the roots are real and repeated equal.

Case 2: If $b^2 - 4ac < 0$ then $\sqrt{b^2-4ac}$ will be imaginary
So, the roots are complex / imaginary and distinct / unequal

Case 3: If $b^2 - 4ac > 0$ then $\sqrt{b^2-4ac}$ will be real.
So, the roots are real and distinct / unequal.

However, If $b^2 - 4ac$ is a perfect square then $\sqrt{b^2-4ac}$ will be rational, and so the roots are rational, otherwise irrational.

Example 1: Discuss the nature of the roots of the following equations:

i) $x^2 + 2x + 3 = 0$ ii) $2x^2 + 5x - 1 = 0$
iii) $2x^2 - 7x + 3 = 0$ iv) $9x^2 - 12x + 4 = 0$

Solution:

i) Comparing $x^2 + 2x + 3 = 0$ with $ax^2 + bx + c = 0$, we have
$a = 1, b = 2, c = 3$
Discriminant (Disc) $= b^2 - 4ac$
$= (2)^2 - 4(1)(3) = 4 - 12 = -8$
$\Rightarrow$ Disc < 0
$\therefore$ The roots are complex / imaginary and distinct / unequal.

ii) Comparing $2x^2 + 5x - 1 = 0$ with $ax^2 + bx + c = 0$, we have
$a = 2, b = 5, c = -1$
Disc $= b^2 - 4ac$
$= (5)^2 - 4(2)(-1)$
$= 25 + 8 = 33$
$\Rightarrow$ Disc > 0 but not a perfect square.
$\therefore$ The roots are irrational and unequal.

iii) Comparing $2x^2 - 7x + 3 = 0$ with $ax^2 + bx + c = 0$ we have
$a = 2, b = -7, c = 3$
Disc $= b^2 - 4ac$
$= (-7)^2 - 4(2)(3)$
$= 49 - 24 = 25 = 5^2$
$\Rightarrow$ Disc > 0 and a perfect square.
$\therefore$ The roots are irrational and unequal.

iv) Comparing $9x^2 - 12x + 4 = 0$ with $ax^2 + bx + c = 0$,we have
$a = 9, b = -12, c = 4$
Disc $= b^2 - 4ac$
$= (-12)^2 - 4(9)(4)$
$= 144 - 144 = 0$
$\Rightarrow$ Disc = 0
$\therefore$ The roots are real and equal.

Example 2: For what values of m will the following equation have equal root? $(m+1)x^2+2(m+3)x+2m+3=0, m\neq -1$

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

$$a=m+1, b=2(m+3), c=2m+3$$

Disc $= b^2 - 4ac$

$$=[2(m+3)]^2-4(m+1)(2m+3)$$

$$=4(m^2+6m+9)-4(2m^2+5m+3)$$

$$= \quad 4m^2 \quad 4m \quad 24$$

The roots of the given equation will be equal, if Disc. = 0 i.e.,

if $\quad -4m^2+4m+24=0$

$\Rightarrow \quad m^2-m-6=0$

$\Rightarrow \quad (m-3)(m+2)=0 \Rightarrow m=3 \text{ or } m=-2$

Hence if m = 3 or m = –2, the roots of the given equation will be equal.

Example 3: Show that the roots of the following equation are real

$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

Also show that the roots will be equal only if $a = b = c$.

Solution: $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

$\Rightarrow \quad x^2-ax-bx+ab+x^2-bx-cx+bc+x^2-cx-ax+ac=0$

$\Rightarrow \quad 3x^2-2(a+b+c)x+ab+bc+ca=0$

Disc $= b^2 - 4ac$

$$=[2(a+b+c)]^2-4(3)(ab+bc+ca)$$

$$=4(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca)$$

$$=4(a^2+b^2+c^2-ab-bc-ca)$$

$$=2(2a^2+2b^2+2c^2-2ab-2bc-2ca)$$

$$=2[a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca]$$

$$=2[(a-b)^2+(b-c)^2+(c-a)^2]$$

= 2(Sum of three squares)

Thus the discriminant cannot be negative.

Hence the roots are real.

The roots will be equal, if the discriminant = 0

This is possible only if $a - b = 0$, $b - c = 0$, $c - a = 0$ i.e., if $a = b = c$.

Exercise 4.7

1. Discuss the nature of the roots of the following equations:

i) $4x^2+6x+1=0$ ii) $x^2-5x+6=0$

iii) $2x^2-5x+1=0$ iv) $25x^2-30x+9=0$

2. Show that the roots of the following equations will be real:

i) $x^2-2\left(m+\dfrac{1}{m}\right)x+3=0; m \bullet 0$

ii) $(b-c)x^2+(c-a)x+(a-b)=0; a,b,c\in Q$

3. Show that the roots of the following equations will be rational:

i) $(p+q)x^2-px-q=0;$ ii) $px^2-(p-q)x-q=0;$

4. For what values of m will the roots of the following equations be equal?

i) $(m+1)x^2+2(m+3)x+m+8 = 0$

ii) $x^2-2(1+3m)x+7(3+2m) = 0$

iii) $(1+m)x^2-2(1+3m)x+(1+8m) = 0$

5. Show that the roots of $x^2+(mx+c)^2=a^2$ will be equal, if $c^2=a^2(1+m^2)$

6. Show that the roots of $(mx + c)^2 = 4ax$ will be equal, if $c=\dfrac{a}{m}; m\neq 0$

7. Prove that $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}$ =1 will have equal roots, if $c^2 = a^2m^2 + b^2$; $a \neq 0$, $b \neq 0$

8. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0$ will be equal, if either $a^3 + b^3 + c^3 = 3abc$ or $b = 0$.

4.11 System of Two Equations Involving Two Variables

We have, so far, been solving quadratic equations in one variable. Now we shall be solving the equations in two variables, when at least one of them is quadratic. To determine the value of two variables, we need a pair of equations.Such a pair of equations is called a **system of simultaneous equations.**

No general rule for the solution of such equations can be laid down except that some how or the other, one of the variables is eliminated and the resulting equation in one variable is solved.

Case I: One Linear Equation and one Quadratic Equation

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following examples:

Example 1: Solve the system of equations:

$x+y=7$ and $x^2-xy+y^2=13$

Solution: $x+y=7 \quad \Rightarrow x=7-y$ (i)

Substituting the value of x in the equation $x^2-xy+y^2=13$ we have

$(7-y)^2-y(7-y)+y^2=13$

$\Rightarrow \quad 49-14y+y^2-7y+y^2+y^2=13$

$\Rightarrow \quad 3y^2-21y+36=0$

$\Rightarrow \quad y^2-7y+12=0$

$\Rightarrow \quad (y-3)(y-4)=0$

$\Rightarrow \quad y=3$ or $y=4$

Putting $y=3$, in (i), we get $x=7-3=4$

Putting $y=4$, in (i), we get $=7-4=3$

Hence solution set = {(4, 3), (3, 4)}.

Note: Two quadratic equations in which xy term is missing and the coefficients of x^2 and y^2 are equal, give a linear equation by subtraction.

Example 2: Solve the following equations:

$x^2+y^2+4x=1$ and $x^2+(y-1)^2=10$

Solution: The given system of equations is

$$\begin{cases} x^2+y^2+4x=1 & \text{(i)} \\ x^2+y^2-2y+1=10 & \text{(ii)} \end{cases}$$

Subtraction gives,

$4x+2y+8=0$

$\Rightarrow 2x+y+4=0$

$\Rightarrow y=-2x-4$ (iii)

Putting the value of y in equation (i),

$x^2+(-2x-4)^2+4x=1 \Rightarrow x^2+4x^2+16x+16+4x=1$

$\Rightarrow \quad 5x^2+20x+15=0 \quad \Rightarrow \quad x\ \ +4x+3=0$

$\Rightarrow \quad (x+3)(x+1)=0 \quad \Rightarrow \quad x=-3$ or $x=-1$

Putting $x=-3$ in (iii), we get; $y=-2(-3)-4=6-4=2$

Putting $x=-1$ in (iii), we get; $y=-2(-1)-4=2-4=-2$

Hence solution set = $\{(-3,\ 2),(-1,\ -2)\}$.

Exercise 4.8

Solve the following systems of equations:

1. $2x-y=4; \quad 2x^2-4xy-y^2=6$ **2.** $x+y=5$; $x^2+2y^2=17$

3. $3x+2y=7; \quad 3x^2=25+2y^2$ **4.** $x+y=5; \dfrac{2}{x}+\dfrac{3}{y}=2, x\neq 0, y\neq 0$

5. $x+y=a+b; \dfrac{a}{x}+\dfrac{b}{y}=2$ **6.** $3x+4y=25; \dfrac{3}{x}+\dfrac{4}{y}=2$

7. $(x-3)^2+y^2=5; \quad 2x=y+6$

8. $(x+3)^2+(y-1)^2=5;\quad x^2+y^2+2x=9$

9. $x^2+(y+1)^2=18;\quad (x+2)^2+y^2=21$

10. $x^2+y^2+6x=1\ ;\quad x^2+y^2+2(x+y)=3$

Case II: Both the Equations are Quadratic in two Variables

The equations in this case are classified as:

i) Both the equations contain only x^2 and y^2 terms.

ii) One of the equations is homogeneous in x and y.

iii) Both the equations are non-homogeneous.

The methods of solving these types of equations are explained through the following examples:

Example 1: Solve the equations: $\begin{cases} x^2+y^2 = 25 \\ 2x^2+3y^2=6 \end{cases}$

Solution: Let $x^2=u$ and $y^2=v$

By this substitution the given equations become

$u+v=25$ (i)

$2u+3v=66$ (ii)

Multiplying both sides of the equation (i) by 2, we have

$2u+2v=50$ (iii)

Subtraction of (iii) from (ii) gives,

$v=16$

Putting the value of v in (i), we have

$u+16=25 \Rightarrow u=9$

$\therefore\quad x^2=9 \Rightarrow x=\pm 3$ and $y^2=16 \Rightarrow y=\pm 4$

Hence solution set = $\{(\pm 3,\ \pm 4)\}$.

Example 2: Solve the equations: $x^2-3xy+2y^2=0; 2x^2-3x+y^2=24$

Solution: The given equations are:

$x^2-3xy+2y^2=0$ (i)

$2x^2-3x+y^2=24$ (ii)

Equation $x^2-3xy+2y^2=0$ is homogeneous in x and y

$\Rightarrow (x-y)(x-2y)=0.$ (Factorizing)

$\Rightarrow x-y=0$ or $x-2y=0$

$\Rightarrow x=y$ ...(iii) $\quad\Rightarrow x=2y$ (iv)

Putting the value of x in (ii), we get

$2y^2-3y+y^2=24$

$\Rightarrow y^2-y-8=0$

$\Rightarrow y=\dfrac{1\pm\sqrt{1+32}}{2}$

$\Rightarrow y=\dfrac{1\pm\sqrt{33}}{2}$

when $y=\dfrac{1+\sqrt{33}}{2}$

from (iii) $x=\dfrac{1+\sqrt{33}}{2}$

when $y=\dfrac{1-\sqrt{33}}{2}$

from (iii) $x=\dfrac{1-\sqrt{33}}{2}$

Putting the value of x in (ii), we get

$2(2y)^2-3(2y)+y^2=24$

$\Rightarrow 8y^2-6y+y^2=24$

$\Rightarrow 3y^2-2y-8=0$

$\Rightarrow (3y+4)(y-2)=0$

$\Rightarrow y=-\dfrac{4}{3},\ 2$

when $y=-\dfrac{4}{3}$,

from (iv) $x=2\left(-\dfrac{4}{3}\right)=\dfrac{8}{3}$

when $y=2$,

from (iv) $x=2(2)=4$

Hence following is the solution set.

$$\left\{\left(\frac{1+\sqrt{33}}{2},\frac{1+\sqrt{33}}{2}\right),\left(\frac{1-\sqrt{33}}{2},\frac{1-\sqrt{33}}{2}\right),\left(-\frac{8}{3},-\frac{4}{3}\right)(4,2)\right\}$$

Example 3: Solve the equations:

$$\begin{cases} x^2-y^2 = 5 \\ 4x^2-3xy=18 \end{cases}$$

Solution Given that $\begin{cases} x^2 - y^2 = 5 & \text{(i)} \\ 4x^2 - 3xy = 18 & \text{(ii)} \end{cases}$

We can get a homogeneous equation in x and y, if we get rid of the constants. For the purpose, we multiply both sides of equation (i) by 18 and both sides of equation (ii) by 5 and get

$$\begin{cases} 18x^2 - 18y^2 = 90 \\ 20x^2 - 15xy = 90 \end{cases}$$

Subtraction gives,

$$2x^2 - 15xy + 18y^2 = 0$$

$$\Rightarrow \quad (x - 6y)(2x - 3y) = 0$$

$$\Rightarrow \quad x - 6y = 0 \text{ or } 2x - 3y = 0$$

Combining each of these equations with any one of the given equations, we can solve them by the method used in the example 1.

$$x - 6y = 0$$

$$\Rightarrow \quad x = 6y$$

$$\because \quad x^2 - y^2 = 5 \qquad \text{from (i)}$$

$$\therefore \quad (6y)^2 - y^2 = 5$$

$$\Rightarrow \quad 35y^2 = 5$$

$$\Rightarrow \quad y^2 = \frac{1}{7}$$

or

$$2x - 3y = 0$$

$$\Rightarrow \quad 2x = 3y \quad \Rightarrow \quad x = \frac{3}{2}y$$

$$\because \quad x^2 - y^2 = 5 \qquad \text{from (i)}$$

$$\therefore \quad \left(\frac{3}{2}y\right)^2 - y^2 = 5$$

$$\Rightarrow \quad 9y^2 - 4y^2 = 20$$

$$\Rightarrow \quad 5y^2 = 20$$

$$\Rightarrow \quad y = \pm\frac{1}{\sqrt{7}}$$

when $y = \frac{1}{\sqrt{7}}$,

when $y = \frac{1}{\sqrt{7}} x \quad 6\left(\frac{-1}{\sqrt{7}}\right) \quad \frac{-6}{\sqrt{7}}$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = \pm 2$$

when $y = 2$,

when $y = -2$

$$x = \frac{3}{2}(-2) = -3$$

Hence Solution set $= \left\{\left(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), \left(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), (3, 2), (3,2)\right\}$

Exercise 4.9

Solve the following systems of Equations:

1. $2x^2 = 6 + 3y^2$; $3x^2 - 5y^2 = 7$

2. $8x^2 = y^2$; $x^2 + 2y^2 = 19$

3. $2x^2 - 8 = 5y^2$; $x^2 - 13 = 2y^2$

4. $x^2 - 5xy + 6y^2 = 0$; $x^2 + y^2 = 45$

5. $12x^2 - 25xy + 12y^2 = 0$; $4x^2 + 7y^2 = 148$

6. $12x^2 - 11xy + 2y^2 = 0$; $2x^2 + 7xy = 60$

7. $x^2 - y^2 = 16$; $xy = 15$

8. $x^2 + xy = 9$; $x^2 - y^2 = 2$

9. $y^2 - 7 = 2xy$; $2x^2 + 3 = xy$

10. $x^2 + y^2 = 5$; $xy = 2$

4.12 Problems on Quadratic Equations

We shall now proceed to solve the problems which, when expressed symbolically, lead to quadratic equations in one or two variables.

In order to solve such problems, we must:

1) Suppose the unknown quantities to be x or y etc.
2) Translate the problem into symbols and form the equations satisfying the given conditions.

Translation into symbolic expression is the main feature of solving problems leading to equations. So, it is always helpful to proceed from concrete to abstract e.g. we may say that:

i) 5 is greater than 3 by 2 = 5 – 3 **ii)** x is greater than 3 by $x - 3$
iii) 5 is greater than y by $5 - y$ **iv)** x is greater than y by $x - y$.

The method of solving the problems will be illustrated through the following examples:

Example 1: Divide 12 into two parts such that the sum of their squares is greater than twice their product by 4.

Solution: Suppose one part = x

$\therefore$ The other part = $12 - x$

Sum of the squares of the parts = $x^2 + (12-x)^2$

twice the product of the parts = $2(x)(12-x)$

By the condition of the question,

$$x^2 + (12-x)^2 - 2x(12-x) = 4$$

$$\Rightarrow \quad x^2 + 144 - 24x + x^2 - 24x + 2x^2 = 4$$

$$\Rightarrow \quad 4x^2 - 48x + 140 = 0 \qquad \Rightarrow \quad x^2 - 12x + 35 = 0$$

$$\Rightarrow \quad (x-5)(x-7) = 0 \qquad \Rightarrow \quad x = 5 \text{ or } x = 7$$

If one part is 5, then the other part = 12 – 5 = 7,
and if one part is 7, then the other part = 12 – 7 = 5

Here both values of x are admissible.
Hence required parts are 5 and 7.

Example 2: A man distributed Rs.1000 equally among destitutes of his street. Had there been 5 more destitutes each one would have received Rs. 10 less. Find the number of destitutes.

Solution: Suppose number of destitutes = x

Total sum = 1000 Rs.

$\therefore$ Each desitute gets = $\frac{1000}{x}$ Rs.

For 5 more destitutes, the number of destitutes would have been $x + 5$

$\therefore$ Each destitute would have got = $\frac{1000}{x+5}$ Rs.

This sum would have been Rs. 10 less than the share of each destitute in the previous case.

$$\therefore \quad \frac{1000}{x+5} = \frac{1000}{x} - 10$$

$$\Rightarrow \quad 1000\,x = 1000\,(x + 5) - 10(x + 5)(x)$$

$$\Rightarrow \quad x^2 + 5x - 500 = 0$$

$$\Rightarrow \quad (x + 25)(x - 20) = 0$$

$$\Rightarrow \quad x = -25 \text{ or } x = 20$$

The number of destitutes cannot be negative. So, –25 is not admissible.
Hence the number of destitutes is 20.

Example 3: The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and the breadth of the room.

Solution: Let the breadth of room = x meters

and the length of room = $x + 3$ meters

$\therefore$ Area of the room = $x(x + 3)$ square meters

By the condition of the question

$x(x + 3) = 180$ (i)

$\Rightarrow$ $x^2 + 3x - 180 = 0$ (ii)

$\Rightarrow$ $(x + 15)(x - 12) = 0$

$\therefore$ $x = -15$ or $x = 12$

As breadth cannot be negative so $x = -15$ is not admissible

$\therefore$ when $x = 12$, we get length $x + 3 = 12 + 3 = 15$

$\therefore$ breadth of the room = 12 meter and length of the room = 15 meter

Example 4: A number consists of two digits whose product is 8. If the digits are interchanged, the resulting number will exceed the original one by 18. Find the number.

Solution : Suppose tens digit = x

and units digit = y

$\therefore$ The number = $10x + y$

By interchanging the digits, the new number = $10y + x$

Product of the digits = xy

By the condition of question;

$xy = 8$ (i)

and $10y + x = 10x + y + 18$ (ii)

Solving (i) and (ii) ;we get

$x = -4$ or $x = 2$.

when $x = -4$, $y = -2$ and when $x = 2$, $y = 4$

Rejecting negative values of the digits,

Tens digit = 2

and Units digit = 4

Hence the required number = 24

Exercise 4.10

1. The product of one less than a certain positive number and two less than three times the number is 14. Find the number.
2. The sum of a positive number and its square is 380. Find the number.
3. Divide 40 into two parts such that the sum of their squares is greater than 2 times their product by 100.
4. The sum of a positive number and its reciprocal is $\frac{26}{5}$ .Find the number.
5. A number exceeds its square root by 56. Find the number.
6. Find two consecutive numbers, whose product is 132.
   (**Hint:** Suppose the numbers are x and $x + 1$).
7. The difference between the cubes of two consecutive even

numbers is 296. Find them.
(**Hint:** Let two consecutive even numbers be x and $x + 2$)

8. A farmer bought some sheep for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

9. A man sold his stock of eggs for Rs. 240. If he had 2 dozen more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

10. A cyclist travelled 48 km at a uniform speed. Had he travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48 km?

11. The area of a rectangular field is 297 square meters. Had it been 3 meters longer and one meter shorter, the area would have been 3 square meters more. Find its length and breadth.

12. The length of a rectangular piece of paper exceeds its breadth by 5 cm. If a strip 0.5 cm wide be cut all around the piece of paper, the area of the remaining part would be 500 square cms. Find its original dimensions.

13. A number consists of two digits whose product is 18. If the digits are interchanged, the new number becomes 27 less than the original number. Find the number.

14. A number consists of two digits whose product is 14. If the digits are interchanged, the resulting number will exceed the original number by 45. Find the number.

15. The area of a right triangle is 210 square meters. If its hypoteneuse is 37 meters long. Find the length of the base and the altitude.

16. The area of a rectangle is 1680 square meters. If its diagonal is 58 meters long, find the length and the breadth of the rectangle.

17. To do a piece of work, A takes 10 days more than B. Together they finish the work in 12 days. How long would B take to finish it alone?
Hint: If some one takes x days to finish a work. The one day's work will be $\frac{1}{x}$.

18. To complete a job, A and B take 4 days working together. A alone takes twice as long as B alone to finish the same job. How long would each one alone take to do the job?

19. An open box is to be made from a square piece of tin by cutting a piece 2 dm square

from each corner and then folding the sides of the remaining piece. If the capacity of the box is to be finish 128 c.dm, find the length of the side of the piece.

20. A man invests Rs. 100,000 in two companies. His total profit is Rs. 3080. If he receives Rs. 1980 from one company and at the rate 1% more from the other, find the amount of each investment.

CHAPTER

8

Mathematical Inductions and Binomial Theorem

8.1 Introduction

Francesco Mourolico (1494-1575) devised the method of induction and applied this device first to prove that the sum of the first n odd positive integers equals n^2. He presented many properties of integers and proved some of these properties using the method of *mathematical induction.*

We are aware of the fact that even one exception or case to a mathematical formula is enough to prove it to be false. Such a case or exception which fails the mathematical formula or statement is called a counter example.

The validity of a formula or statement depending on a variable belonging to a certain set is established if it is true for each element of the set under consideration.

For example, we consider the statement $S(n) = n^2 - n + 41$ is a prime number for every natural number n. The values of the expression $n^2 - n + 41$ for some first natural numbers are given in the table as shown below:

n	1	2	3	4	5	6	7	8	9	10	11
$S(n)$	41	43	47	53	61	71	83	91	113	131	151

From the table, it appears that the statement $S(n)$ has enough chance of being true. If we go on trying for the next natural numbers, we find $n = 41$as a counter example which fails the claim of the above statement. So we conclude that to derive a general formula without proof from some special cases is not a wise step. This example was discovered by Euler (1707-1783).

Now we consider another example and try to formulate the result. Our task is to find the sum of the first n odd natural numbers. We write first few sums to see the pattern of sums.

n (The number of terms)	Sum
1	$1= 1^2$
2	$1+3= 4 = 2^2$
3	$1+3+5= 9=3^2$
4	$1+3+5+7=16=4^2$
5	$1+3+5+7+9=25 = 5^2$
6	$1+3+5+7+9+11=36= 6^2$

The sequence of sums is $(1)^2$, $(2)^2$, $(3)^2$, $(4)^2$, ...

We see that each sum is the square of the number of terms in the sum. So the following statement seems to be true.

For each natural number n,

$1 + 3 + 5 + \dots + (2n - 1) = n^2$ . . . . (i) $\quad$ ($\because$ nth term$=1+(n-1)2$)

But it is not possible to verify the statement (i) for each positive integer n, because it involves infinitely many calculations which never end.

The method of mathematical induction is used to avoid such situations.Usually it is used to prove the statements or formulae relating to the set {1,2,3,...} but in some cases, it is also used to prove the statements relating to the set (0 ,1,2 ,3,...}.

8.2 Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer n is such that

1) $S(1)$ is true i.e., $S(n)$ is true for $n = 1$ and
2) $S(k + 1)$ is true whenever $S(k)$ is true for any positive integer k, then $S(n)$ is true for all positive integers.

Procedure:

1. Substituting $n = 1$, show that the statement is true for $n = 1$.
2. Assuming that the statement is true for any positive integer k, then show that it is true for the next higher integer. For the second condition, one of the following two methods can be used:

M_1 Starting with one side of $S(k +1)$, its other side is derived by using $S(k)$.

M_2 $S(k + 1)$ is established by performing algebraic operations on $S(k)$.

Example 1: Use mathematical induction to prove that $3+6+9+\dots+3n=\frac{3n(n+1)}{2}$ for every positive integer n.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n): \quad 3+6+9....+3n=\frac{3n(n+1)}{2} \qquad \text{(i)}$$

1. When n = 1, S(1) becomes

$$S(1): 3=\frac{3(1)(1+1)}{2}=3$$

Thus S(1) is true i.e., condition (1) is satisfied.

2. Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$3+6+9....+3k=\frac{3k(k+1)}{2} \qquad \text{(A)}$$

The statement for n = k+ 1 becomes

$$3+6+9....+3k+3(k+1)=\frac{3k(k+1)\left[(k+1)+1\right]}{2}$$
$$=\frac{3(k+1)(k+2)}{2} \qquad \text{(B)}$$

Adding 3(k+ 1) on both the sides of (A) gives

$$3+6+9+....+3k+3(k+1)=\frac{3k(k+1)}{2}+3(k+1)$$
$$=3(k+1)(\frac{k}{2}+1)$$
$$=\frac{3(k+1)(k+2)}{2}$$

Thus $S(k + 1)$ is true if $S(k)$ is true, so the condition (2) is satisfied.

Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

Example 2: Use mathematical induction to prove that for any positive integer n,

$$1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

Solution: Let $S(n)$ be the given statement,

$$S(n): \quad 1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

1. If n = 1, S (1) becomes

$$S(1):(1)^2=\frac{1(1+1)(2\times1+1)}{6}=\frac{1\times2\times3}{6}=1$$

Thus S(1) is true, i.e., condition (1) is satisfied.

2. Let us assume that $S(k)$ is true for any $k \in N$, that is,

$$1^2+2^2+3^2+....+k^2=\frac{k(k+1)(2k+1)}{2} \qquad \text{(A)}$$

$$S(k+1):\ 1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+1)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6} \qquad \text{(B)}$$

Adding $(k+1)^2$ to both the sides of equation (A), we have

$$1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$
$$=\frac{(k+1)(2k^2+k+6k+6)}{6}$$
$$=\frac{(k+1)(2k^2+7k+6)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

Thus the condition (2) is satisfied. Since both the conditions are satisfied, therefore, by mathematical induction, the given statement holds for all positive integers.

Example 3: Show that $\frac{n^3+2n}{3}$ represents an integer $\forall n \in N$.

Solution: Let $S(n)=\frac{n^3+2n}{3}$

1. When n = 1, S(1) becomes

$$S(1)=\frac{1^3+2(1)}{3}=\frac{3}{3}=1\in Z$$

2. Let us assume that $S(n)$ is ture for any $n = k \in W$, that is,

$S(k)=\frac{k^3+2k}{3}$ represents an integer.

Now we want to show that $S(k+1)$ is also an integer. For $n=k+1$, the statement becomes

$$S(k+1)=\frac{(k+1)^3+2(k+1)}{3}$$
$$=\frac{k^3+3k^2+3k+1+2k+2}{3}=\frac{(k^3+2k)+(3k^2+3k+3)}{3}$$
$$=\frac{(k^3+2k)+3(k^2+k+1)}{3}$$
$$=\frac{k^3+2k}{3}+(k^2+k+1)$$

As $\frac{k^3+2k}{3}$ is an integer by assumption and we know that $(k^2 + k + 1)$is an integer as $k \in W$.

$S(k + 1)$ being sum of integers is an integer, thus the condition (2) is satisfied.

Since both the conditions are satisfied, therefore, we conclude by mathematical induction that $\frac{n^3+2n}{3}$ represents an integer for all positive integral values of n.

Example 4: Use mathematical induction to prove that

$3+3.5+3.5^2+....+3.5^n=\frac{3(5^{n+1}-1)}{4}$ whenever n is non-negative integer.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n):\quad 3+3.5+3.5^2+....+3.5^n=\frac{3(5^{n+1}-1)}{4}$$

The dot (.)between two number, stands, for multipication symbol.

1. For n= 0, S(0) becomes $S(0):3.5^0=\frac{3(5^{0+1}-1)}{4}$ or $3=\frac{3(5-1)}{4}=3$

Thus S(0) is true i.e., conditions (1) is satisfied.

2. Let us assume that $S(k)$ is true for any $k \in W$, that is,

$$S(k):3+3.5+3.5^2+....+3.5^k=\frac{3(5^{k+1}-1)}{4} \qquad \text{(A)}$$

Here $S(k+1)$ becomes

$$S(k+1):3+3.5+3.5^2+....+3.5^k+3.5^{k+1}=\frac{3(5^{(k+1)+1}-1)}{4}$$
$$=\frac{3(5^{k+2}-1)}{4} \qquad \text{(B)}$$

Adding 3.5^{k+1} on both sides of (A), we get

$$3+3.5+3.5^2+....+3.5^k+3.5^{k+1}=\frac{3(5^{k+1}-1)}{4}+3.5^{k+1}$$
$$=\frac{3(5^{k+1}-1+4.5^{k+1})}{4}$$
$$=\frac{3[5^{k+1}(1+4)-1]}{4}$$
$$=\frac{3(5^{k+2}-1)}{4}$$

This shows that $S(k + 1)$ is true when $S(k)$ is true. Since both the conditions are satisfied, therefore, by the principle of mathematical induction, $S(n)$ in true for each $n \in W$.

Care should be taken while applying this method. Both the conditions (1) and (2) of the principle of mathematical induction are essential. The condition (1) gives us a starting point but the condition (2) enables us to proceed from one positive integer to the next. In the condition (2) we do not prove that $S(k+1)$ is true but prove only that if $S(k)$is true, then $S(k+1)$ is true. We can say that any proposition or statement for which only one condition is satisfied, will not be truefor all n belonging to the set of positive integers.

For example, we consider the statement that 3^n is an even integer for any positive integer n. Let $S(n)$ be the given statement.

Assume that $S(k)$ is true, that is, 3^k in an even integer for $n = k$. When 3^k is even, then $3^k + 3^k + 3^k$ is even which implies that $3^k.3 = 3^{k+1}$ is even.

This shows that $S(k+1)$ will be true when $S(k)$ is true. But 3^1 is not an even integer which reflects that the first condition does not hold. Thus our supposition is false.

Note:- There is no integer n for which 3^n is even.

Sometimes, we wish to prove formulae or statements which are true for all integers n greater than or equal to some integer i, where $i \neq 1$. In such cases, $S(1)$ is replaced by $S(i)$ and the condition (2) remains the same. To tackle such situations, we use the principle of extended mathematical induction which is stated as below:

8.3 Principle of Extended Mathematical Induction

Let i be an integer. If a formula or statement $S(n)$ for $n \geq i$ is such that

1) $S(i)$ is true and
2) $S(k+1)$ is true whenever $S(k)$ is true for any integer $n \geq i$.

Then $S(n)$ is true for all integers $n \geq i$.

Example 5: Show that $1 + 3 + 5 + \ldots (2n+5) = (n+3)^2$ for integral values of $n \geq -2$.

Solution:

1. Let $S(n)$ be the given statement, then for $n = -2$, $S(-2)$ becomes, $2(-2)+5=(-2+3)^2$, i.e., $1 = (1)^2$ which is true.
   Thus $S(-2)$ is true i.e., the condition (1) is satisfied
2. Let the equation be true for any $n = k \in Z$, $k \geq -2$, so that
   $1+3+5+\ldots+(2k+5) = (k+3)^2$ (A)

$S(k+1): 1+3+5+\ldots+(2k+5)+(2\overline{k+1}+5) = (\overline{k+1}+3)^2 = (k+4)^2$ (B)

Adding $(2\overline{k+1}+5) = (2k+7)$ on both sides of equation (A) we get,

$$\begin{aligned} 1+3+5+\ldots+(2k+5)+(2k+7) &= (k+3)^2+(2k+7) \\ &= k^2+6k+9+2k+7 \\ &= k^2+8k+16 \\ &= (k+4)^2 \end{aligned}$$

Thus the condition (2) is satisfied. As both the conditions are satisfied, so we conclude that the equation is true for all integers $n \geq -2$.

Example 6: Show that the inequality $4^n > 3^n + 4$ is true, for integral values of $n \geq 2$.

Solution: Let $S(n)$ represents the given statement i.e., $S(n)$: $4^n > 3^n + 4$ for integral values of $n \geq 2$.

1. For $n = 2$, $S(2)$ becomes
   $S(2)$: $4^2 > 3^2 + 4$, i.e., $16 > 13$ which is true.
   Thus $S(2)$ is true, i.e., the first condition is satisfied.
2. Let the statement be true for any $n = k(\geq 2) \in Z$, that is
   $4^k > 3^k + 4$ (A)
   Multiplying both sides of inequality (A) by 4, we get
   or $4.4^k > 4(3^k + 4)$
   or $4^{k+1} > (3+1)3^k + 16$
   or $4^{k+1} > 3^{k+1} + 4 + 3^k + 12$
   or $4^{k+1} > 3^{k+1} + 4$ $(\because 3^k + 12 > 0)$ (B)
   The inequality (B), satisfies the condition (2).
   Since both the conditions are satisfied, therefore, by the principle of extended mathematical induction, the given inequality is true for all integers $n \geq 2$.

Exercise 8.1

Use mathematical induction to prove the following formulae for every positive integer n.

1. $1+5+9+\ldots+(4n-3) = n(2n-1)$

2. $1+3+5+....+(2n-1)=n^2$

3. $1+4+7+....+(3n-2)=\frac{n(3n-1)}{2}$

4. $1+2+4+....+2^{n-1}=2^n-1$

5. $1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]$

6. $2+4+6+....+2n=n(n+1)$

7. $2+6+18+....+2\times 3^{n-1}=3^n-1$

8. $1\times 3+2\times 5+3\times 7+....+n\times(2n+1)=\frac{n(n+1)(4n+5)}{6}$

9. $1\times 2+2\times 3+3\times 4+....+n\times(n+1)=\frac{n(n+1)(n+2)}{3}$

10. $1\times 2+3\times 4+5\times 6+....+(2n-1)\times 2n=\frac{n(n+1)(4n-1)}{3}$

11. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+....+\frac{1}{n(n+1)}=1 \quad \frac{1}{n+1}$

12. $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

13. $\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+....+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$

14. $r+r^2+r^3+....+r^n=\frac{r(1-r^n)}{1-r}, \qquad (r \neq 1)$

15. $a+(a+d)+(a+2d)+....+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$

16. $1.\underline{|1}+2\underline{|2}+3\underline{|3}+....+n\underline{|n}=\underline{|n+1}-1$

17. $a_n=a_1+(n-1)d$ when, a_1, a_1+d, a_1+2d, ... form an A.P.

18. $a_n=a_1r^{n-1}$ when a_1, a_1r, a_1r^2,... form a G.P.

19. $1^2+3^2+5^2+....+(2n-1)^2=\frac{n(4n^2-1)}{3}$

20. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+....+\binom{n+2}{3}=\binom{n+3}{4}$

21. Prove by mathematical induction that for all positive integral values of n

i) n^2+n is divisible by 2. ii) 5^n-2^n is divisible by 3.

iii) 5^n-1 is divisible by 4. iv) $8\times 10^n-2$ is divisible by 6.

v) n^3-n is divisible by 6.

22. $\frac{1}{3}+\frac{1}{3^2}+....+\frac{1}{3^n}=\frac{1}{2}\left[1-\frac{1}{3^n}\right]$

23. $1^2-2^2+3^2-4^2+....+(-1)^{n-1}\ .n^2=\frac{(-1)^{n-1}.n(n+1)}{2}$

24. $1^3+3^3+5^3+....+(2n-1)^3=n^2[2n^2-1]$

25. $x+1$ is a factor of $x^{2n}-1; (x\neq -1)$

26. $x-y$ is a factor of x^n-y^n; $(x\neq y)$

27. $x+y$ is a factor of $x^{2n-1}+y^{2n-1}$ $(x\neq -y)$

28. Use mathematical induction to show that
$1+2+2^2+....+2^n=2^{n+1}-1$ for all non-negative integers n.

29. If A and B are square matrices and $AB=BA$, then show by mathematical induction that $AB^n=B^nA$ for any positive integer n.

30. Prove by the Principle of mathematical induction that n^2-1 is divisible by 8 when n is an odd positive integer.

31. Use the principle of mathematical induction to prove that $\ln x^n = n\ln x$ for any integer $n\geq 0$ if x is a positive number. Use the principle of extended mathematical induction to prove that:

32. $n! > 2^n-1$ for integral values of $n\geq 4$.

33. $n^2 > n+3$ for integral values of $n\geq 3$.

34. $4^n > 3^n+2^{n-1}$ for integral values of $n\geq 2$.

35. $3^n < n!$ for integral values of $n > 6$.

36. $n! > n^2$ for integral values of $n\geq 4$.

37. $3 + 5 + 7 + ... + (2n + 5) = (n + 2)(n + 4)$ for integral values of $n \geq -1$.

38. $1 + nx \leq (1 + x)^n$ for $n \geq 2$ and $x > -1$

8.4 Binomial Theorem

An algebraic expression consisting of two terms such as $a + x$, $x - 2y$, $ax + b$ etc., is called a binomial or a binomial expression.

We know by actual multiplication that

$(a + x)^2 = a^2 + 2ax + x^2$ (i)

$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$ (ii)

The right sides of (i) and (ii) are called binomial expansions of the binomial $a + x$ for the indices 2 and 3 respectively.

In general, the rule or formula for expansion of a binomial raised to any positive integral power n is called the binomial theorem for positive integral index n.For any positive integer n,

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + ... + \binom{n}{r-1}a^{n-(r-1)}x^{r-1} \quad \text{(A)}$$

$$+\binom{n}{r}a^{n-r}x^r + ... + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n$$

or briefly

$$(a+x)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}x^r$$

where a and x are real numbers.

The rule of expansion given above is called the binomial theorem and it also holds if a or x is complex.

Now we prove the Binomial theorem for any positive integer n, using the principle of mathematical induction.

Proof: Let $S(n)$ be the statement given above as (A).

1. If $n = 1$, we obtain

$$S(1): (a+x)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}x = a + x$$

Thus condition (1) is satisfied.

2. Let us assume that the statement is true for any $n = k \in N$, then

$$(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + ... + \binom{k}{r-1}a^{k-(r-1)}x^{r-1} \quad \binom{k}{r}a^{k-r}x^r$$

$$+ .... + \binom{k}{k}ax^k + \binom{k}{k}x^k \quad \text{(B)}$$

$$S(k+1): (a+k)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k \times x + \binom{k+1}{2}a^{k-1} \times x^2 + ... +$$

$$\binom{k+1}{r-1}a^{k-r+2} \times x^{r-1} + \binom{k+1}{r}a^{k-r+1} \times x^r + ... + \binom{k+1}{k}a \times x^k + \binom{k+1}{k+1}x^{k+1} \quad \text{(C)}$$

Multiplying both sides of equation (B) by $(a+x)$, we have

$$(a+x)(a+x)^k = (a+x)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + ... + \binom{k}{r-1}a^{k-r+1}x^{r-1}\right.$$

$$\left. + \binom{k}{r}a^{k-r} x^r + ... + \binom{k}{k-1}ax^{k-1} + \binom{k}{k}x^k\right]$$

$$= \left[\binom{k}{0}a^{k+1} + \binom{k}{1}a^k x + \binom{k}{2}a^{k-1}x^2 + ... + \binom{k}{r-1}a^{k-r+2}x^{r-1}\right.$$

$$\left. + \binom{k}{r}a^{k-r+1}x^r + ... + \binom{k}{k-1}a^2x^{k-1} + \binom{k}{k}ax^k\right]$$

$$+\left[\binom{k}{0}a^k x+\binom{k}{1}a^{k-1}x^2+\binom{k}{2}a^{k-2}\times x^3+\ldots.+\binom{k}{r-1}a^{k-r+1}\times x^r\right.$$

$$\left.+\binom{k}{r}a^{k-r}x^{r+1}+\ldots.+\binom{k}{k-1}ax^k+\binom{k}{k}x^{k+1}\right]$$

$$=\binom{k}{0}a^{k+1}+\left[\binom{k}{1}+\binom{k}{0}\right]a^k x+\left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}x^2+\ldots.$$

$$+\left[\binom{k}{r}+\binom{k}{r-1}\right]a^{k-r+1}x^r+\ldots.+\left[\binom{k}{k}+\binom{k}{k-1}\right]ax^k+\binom{k}{k}x^{k+1}$$

As $\binom{k}{0}=\binom{k+1}{0},\binom{k}{k}=\binom{k+1}{k+1}\text{and}\binom{k}{r}+\binom{k}{r-1}=\binom{k+1}{r}\text{for}1\le r\le k$

$$\therefore (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^k x+\binom{k+1}{2}a^{k-1}x^2+\ldots.$$

$$+\binom{k+1}{r}a^{k-r+1}x^r+\ldots.+\binom{k+1}{k}a\ x^k+\binom{k+1}{k+1}x^{k+1}$$

We find that if the statement is true of n = k, then it is also true for n = k +1. Hence we conclude that the statement is true for all positive integral values of n.

Note: $\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n}$ are called the binomial coefficients.

The following points can be observed in the expansion of (a + x)^n

1. **The number of terms in the expansion is one greater than its index.**
2. **The sum of exponents of a and x in each term of the expansion is equal to its index.**
3. **The exponent of a decreases from index to zero.**
4. **The exponent of x increases from zero to index.**
5. **The coefficients of the terms equidistant from beginning and end of the expansion are equal as** $\binom{n}{r}=\binom{n}{n-r}$
6. **The (r + l)th term in the expansion** $\binom{n}{r}a^{n-r}x^r$ **and we denote it as** T_{r+1} **i.e.,**

$$T \qquad \binom{}{}a^{n\ \ r}x^r$$

As all the terms of the expansion can be got from it by putting r = 0, 1,2..... n, so we call it as the **general term** of the expansion.

Example 1: Expand $\left(\frac{a}{2}-\frac{2}{a}\right)^6$ also find its general term.

Solution: $\left(\frac{a}{2}-\frac{2}{a}\right)^6=\left(\frac{a}{2}+\left(\frac{-2}{a}\right)\right)^6$

$$=\left(\frac{a}{2}\right)^6+\binom{6}{1}\left(\frac{a}{2}\right)^5\left(-\frac{2}{a}\right)+\binom{6}{2}\left(\frac{a}{2}\right)^4+\left(-\frac{2}{a}\right)^2+\binom{6}{3}\left(\frac{a}{2}\right)^3\left(-\frac{2}{a}\right)^3$$

$$+\binom{6}{4}\left(\frac{a}{2}\right)^2\left(-\frac{2}{a}\right)^4+\binom{6}{5}\left(\frac{a}{2}\right)\left(-\frac{2}{a}\right)^5+\left(-\frac{2}{a}\right)^6$$

$$=\frac{a^6}{64}+6\left(\frac{a^5}{32}\right)\left(-\frac{2}{a}\right)+\frac{6.5}{2.1}.\frac{a^4}{16}.\frac{4}{a^2}.+\frac{6.5.4}{3.2.1}.\frac{a^3}{8}.\left(-\frac{8}{a^3}\right)+\frac{6.5}{2.1}.\frac{a^2}{4}.\frac{16}{a^4}$$

$$+6.\frac{a}{2}\left(\frac{-32}{a^5}\right)+\frac{64}{a^6}$$

$$=\frac{a^6}{64}-\frac{3}{8}a^4+\frac{15}{4}a^2-20+\frac{60}{a^2}-\frac{96}{a^4}+\frac{64}{a^6}$$

T_{r+1}, the general term is given by

$$T_{r+1}=\binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\left(-\frac{2}{a}\right)^r=\binom{6}{r}\frac{a^{6-r}}{2^{6-r}}(-1)^r\frac{2^r}{a^r}$$

$$=(-1)^r\binom{6}{r}\frac{a^{6-r}.a^{-r}}{2^{6-r}.2^{-r}}=(-1)^r.\binom{6}{r}\frac{a^{6-2r}}{2^{6-2r}}=(-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-2r}$$

Example 2: Evaluate $(9.9)^5$

Solution: $(9.9)^5 = (10-.1)^5$

$= (10)^5 + 5 \times (10)4 \times (-.1) + 10(10)^3 \times (-.1)^2 + 10(10)2 \times (-.1)^3 + 5(10)(-.1)^4 + (-.1)^5$

$= 100000 - (.5)(10000) + (10000X.01) + 1000(-.001) + 50(.0001) - .00001$

$= 100000 - 5000 + 100 - 1 + .005 - .00001$

$= 100100.005 - 5001.00001$

$= 95099.00499$

Example 3: Find the specified term in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$;

i) the term involving x^5 ii) the fifth term

iii) the sixth term from the end. iv) coefficient of term involving x^{-1}

Solution:

i) Let T_{r+1} be the term involving x^5 in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$, then

$$T_{r+1}=\binom{11}{r}\left(\frac{3}{2}x\right)^{11-r}\left(-\frac{1}{3x}\right)^r=\binom{11}{r}\frac{3^{11-r}}{2^{11-r}}x^{11-r}.(-1)^r.3^{-r}.x^{-r}$$

$$=(-1)^r.\binom{11}{r}\frac{3^{11-2r}}{2^{11-r}}.x^{11-2r}$$

As this term involves x^5 , so the exponent of x is 5, that is,

$$11-2r=5$$

or $\quad -2r=5-11 \Rightarrow r=3$

Thus T_4 involves x^5

$$\therefore\ T_4=(-1)^3.\binom{11}{3}\frac{3^{11-6}}{2^{11-3}}.x^{11-6}=(-1).\frac{11.10.9}{3.2.1}.\frac{3^5}{2^8}.x^5$$

$$=\frac{165\times 243}{256}x^5 \qquad -\frac{40095}{256}x^5$$

ii) Putting **r** = 4 in T_{r+1}, we get T_5

$$\therefore\ T_5=(-1)^4\binom{11}{4}\frac{3^{11-8}}{2^{11-4}}.x^{11-8}=\frac{11.10.9.8}{4.3.2.1}.\frac{3^3}{2^7}.x^3$$

$$=\frac{11\times 10\times 3}{1}.\frac{27}{128}x^3=\frac{165\times 27}{64}x^3$$

$$=\frac{4455}{64}x^3$$

iii) The 6th term from the end term will have (11 + 1) – 6 i.e., 6 terms before it,

$\therefore$ It will be (6 + 1) th term i.e.. the 7th term of the expansion.

Thus $\quad T_7=(-1)^6\binom{11}{6}\frac{3^{11-12}}{2^{11-6}}x^{11-12}=\frac{11.10.9.8.7}{5.4.3.2.1}\frac{3^{-1}}{2^5}x^{-1}$

$$=\frac{11\times 6\times 7}{1}.\frac{1}{3\times 32}.\frac{1}{x}=\frac{77}{16x}$$

iv) $\frac{77}{16}$ is the coefficient of the term involving x^{-1}

8.3.1 The Middle Term in the Expansion of $(a + x)^n$

In the expansion of $(a + x)^n$, the total number of terms is $n + 1$

Case I: (n is even)

If n is even then n+1 is odd, so $\left(\frac{n+1}{2}\right)$th term will be the only one middle term in the expansion.

Case II: (n is odd)

If n is odd then $n + 1$ is even so $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms of the expansion will be the two middle terms.

Example 4: Find the following in the expansion of $\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$;

i) the term independent of x. ii) the middle term

Solution: i) Let T_{r+1} be the term independent of x in the expansion of

$\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$, then

$$T_{r+1} == \binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\left(\frac{2}{x^2}\right)^{r} \quad \binom{12}{r}\frac{x^{12-r}}{2^{12-r}}.2^r.x^{-2r}$$

$$=\binom{12}{r}2^{2r-12}.x^{12-3r}$$

As the term is independent of x, so exponent of x, will be zero.

That is, $12 - 3r = 0 \Rightarrow r = 4$.

Therefore the required term $T_5 = \binom{12}{4}2^{8-12}.x^{12-12}$

$$=\frac{12\times11\times10\times9}{4\times3\times2\times1}.2^{-4}.x^{0}$$

$$=\frac{11\times45}{2^4}=\frac{495}{16}$$

ii) In this case, $n = 12$ which is even, so

$\therefore$ $\left(\frac{12}{2}+1\right)$th term is the middle term in the expansion,

i.e., T_7 is the required term.

$$T_7 = \binom{12}{6}\left(\frac{x}{2}\right)^{12-6}.\left(\frac{2}{x^2}\right)^{6}$$

$$== \binom{12}{6}\frac{x^6}{2^6}.\frac{2^6}{x^{12}} \quad \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1}.x^{6-12}$$

$$=\frac{12\times11\times7}{x^6}=\frac{924}{x^6}$$

8.3.2 Some Deductions from the binomial expansion of $(a + x)^n$.

We know that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots$$

$$+\binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n \qquad \text{(I)}$$

(i) If we put a = 1, in (I), then we have;

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{r}x^r + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \qquad \text{(II)}$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r + ... + nx^{n-1} + x^n$$

$$\left(\because \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)....(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)....(n-r+1)}{r!} \right)$$

ii) Putting a = 1 and replacing x by $-x$, in (I), we get.

$$(1-x)^n = \binom{n}{0} + \binom{n}{1}(-x) + \binom{n}{2}(-x)^2 + \binom{n}{3}(-x)^3 + ... + \binom{n}{n-1}(-x)^{n-1} + \binom{n}{n}(-x)^n$$

$$= \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + ... + (-1)^{n-1}\binom{n}{n-1}x^{n-1} + (-1)^n\binom{n}{n}x^n .... \qquad \text{(III)}$$

iii) We can find the sum of the binomial cofficients by putting a = 1and x = 1 in (I).

i.e., $$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n}$$

or $$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n}$$

Thus the sum of coefficients in the binomial expansion equals to 2^n .

iv) Putting a = 1 and x = –1, in (i) we have

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n}$$

Thus $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0$$

If n is odd positive integer, then

$$\binom{n}{0} + \binom{n}{2} + ... + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + ... + \binom{n}{n}$$

If n is even positive integer, then

$$\binom{n}{0} + \binom{n}{2} + ... + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + ... + \binom{n}{n-1}$$

Thus sum of odd coefficients of a binomial expansion equals to the sum of its even coefficients.

Example 5: Show that: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} = n.2^{n-1}$

Solution:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} = n + 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + ... + n.1$$

$$= n.\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + ... + 1\right]$$

$$= n.\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1}\right]$$

$$= n.2^{n-1}$$

Exercise 8.2

1. Using binomial theorem, expand the following:

i) $(a+2b)^5$ ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$ iii) $\left(3a-\frac{x}{3a}\right)^4$

iv) $\left(2a-\frac{x}{a}\right)^7$ v) $\left(\frac{x}{2y}+\frac{2y}{x}\right)^8$ vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

2. Calculate the following by means of binomial theorem:

i) $(0.97)^3$ ii) $(2.02)^4$ iii) (9.98) iv) $(21)^5$

3. Expand and simplify the following:

i) $\left(a+\sqrt{2}x\right)^4+\left(a-\sqrt{2}x\right)^4$ ii) $\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5$

iii) $(2+i)^5-(2-i)^5$ iv) $\left(x+\sqrt{x^2-1}\right)^3+\left(x-\sqrt{x^2-1}\right)^3$

4. Expand the following in ascending power of x:

i) $(2+x-x^2)^4$ ii) $(1-x+x^2)^4$ iii) $(1-x-x^2)^4$

5. Expand the following in descending powers of x:

i) $(x^2+x-1)^3$ ii) $\left(x-1-\frac{1}{x}\right)^3$

6. Find the term involving:

i) x^4 in the expansion of $(3-2x)^7$

ii) x^{-2} in the expansion of $\left(x-\frac{2}{x^2}\right)^{13}$

iii) a^4 in the expansion of $\left(\frac{2}{x}-a\right)^9$

iv) y^3 in the expansion of $\left(x-\sqrt{y}\right)^{11}$

7. Find the coefficient of;

i) x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

ii) x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

8. Find 6th term in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

9. Find the term independent of x in the following expansions.

i) $\left(x-\frac{2}{x}\right)^{10}$ ii) $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$ iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4$

10. Determine the middle term in the following expansions:

i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$ ii) $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$ iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

11. Find $(2n+1)$ th term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$

12. Show that the middle term of $(1+x)^{2n}$ is $=\frac{1.3.5....(2n-1)}{n!}2^n x^n$

13. Show that: $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+....+\binom{n}{n-1}=2^{n-1}$

14. Show that: $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+....+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

8.4 The Binomial Theorem when the index *n* is a negative integer or a fraction.

When *n* is a negative integer or a fraction, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$+\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots\dots$$

provided $|x|<1$.

The series of the type

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

is called the binomial series.

Note (1): The proof of this theorem is beyond the scope of this book.

(2) Symbols $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}$ etc are meaningless when n is a negative integer or a fraction.

(3) The general term in the expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$$

Example 1: Find the general term in the expansion of $(1+x)^{-3}$ when $|x|<1$

Solution: $T_{r+1} = \frac{(-3)(-4)(-5)\dots(-3-r+1)}{r!}x^r$

$$= \frac{(-1)^r.3.4.5\dots(r+2)}{r!}x^r$$

$$= (-1)^r.\frac{1.2.3.4.5\dots(r+2)}{1.2.r!}x^r$$

$$= (-1)^r\frac{r!.(r+1)(r+2)}{2.r!}x^r$$

$$= (-1)^r.\frac{(r+1)(r+2)}{2}x^r$$

Some particular cases of the expansion of $(1+x)^n$, $n<0$.

i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots + (-1)^r x^r + \dots$

ii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots + (-1)^r(r+1)x^r + \dots$

iii) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + (-1)^r\frac{(r+1)(r+2)}{2}x^r + \dots$

iv) $(1-x)^{-1} = 1 + x + x^2 - x^3 + \dots\dots + x^r + \dots$

v) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots + (r+1)x^r + \dots$

vi) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{2}x^r + \dots$

8.5 Application of the Binomial Theorem

Approximations: We have seen in the particular cases of the expansion of $(1+x)^n$ that the power of x go on increasing in each expansion. Since $|x| < 1$, so

$$|x|^r < |x| \text{ for } r = 2,3,4\dots$$

This fact shows that terms in each expansion go on decreasing numerically if $|x| < 1$. Thus some initial terms of the binomial series are enough for determining the approximate values of binomial expansions having indices as negative integers or fractions.

Summation of infinite series: The binomial series are conveniently used for summation of infinite series..The series (*whose sum is required*) is compared with

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

to find out the values of *n* and x. Then the sum is calculated by putting the values of *n* and x in $(1+x)^n$.

Example 2: Expand $(1-2x)^{1/3}$ to four terms and apply it to evaluate $(.8)^{1/3}$ correct to three places of decimal.

Solution: This expansion is valid only if $|2x|<1$ or $2|x|<1$ or $|x|<-$, that is

$$(1-2x)^{1/3}=1+\frac{1}{3}(-2x)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-2x)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-2x)^3+\ldots\ldots$$

$$=1-\frac{2}{3}x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2.1}(4x^2)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3.2.1}(-8x^3)-\ldots.$$

$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{1.2.5}{3.3.3}.\frac{1}{3.2.1}(8x^3)-\ldots$$

$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{40}{81}x^3-\ldots.$$

Putting $x=.1$ in the above expansion we have

$$(1-2(.1))^{1/3}=1-\frac{2}{3}(.1)-\frac{4}{9}(.1)^2-\frac{40}{81}(.1)^3-\ldots$$

$$=1-\frac{.2}{3}-\frac{.04}{9}-\frac{.04}{81}\ldots \qquad (\because 40\times .001=.04)$$

$$\approx 1-.06666-.00444-.00049=1-.07159=.92841$$

Thus $(.8)^{1/3}\approx .928$

Alternative method:

$$(.8)^{1/3}=(1-.2)^{1/3}=1-\frac{.2}{3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-.2)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-.2)^3+\ldots$$

Simplify onward by yourself.

Example 3: Expand $(8-5x)^{-2/3}$ to four terms.

Solution: $(8-5x)^{-2/3}=\left(8\left(1-\frac{5x}{8}\right)\right)^{-\frac{2}{3}}=8^{-\frac{2}{3}}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}=(8^{\frac{1}{3}})^{-2}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}$

$$=\frac{1}{4}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}$$

$$=\frac{1}{4}\left[1+\left(-\frac{2}{3}\right)\left(-\frac{5}{8}x\right)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2!}\left(-\frac{5}{8}x\right)^2+\frac{\left(-\frac{2}{3}\right)\left(\frac{-5}{3}\right)\left(-\frac{8}{3}\right)}{3!}\left(-\frac{5}{8}x\right)^3+\ldots\right]$$

$$=\frac{1}{4}\left[1+\frac{5}{12}x+\frac{5}{9}\times\frac{25}{64}x^2+\frac{40}{81}\times\frac{125}{8\times 64}x^3+\ldots\right]$$

$$=\frac{1}{4}+\frac{5}{48}x+\frac{125}{2304}x^2+\frac{625}{20736}x^3+\ldots.$$

The expansion of $\left(1-\frac{5}{8}x\right)^{-2/3}$ is valid when $\left|\frac{5}{8}x\right|<1$

or $\frac{5}{8}|x|<1 \Rightarrow |x|<\frac{8}{5}$

Example 4: Evaluate $\sqrt[3]{30}$ correct to three places of decimal.

Solution: $\sqrt[3]{30}=(30)^{1/3}=(27+3)^{\frac{1}{3}}$

$$=\left[27\left(1+\frac{3}{27}\right)\right]^{1/3}=(27)^{1/3}\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left[1+\frac{1}{3}\cdot\frac{1}{9}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{9}\right)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{9}\right)^3+\ldots\right]$$

$$=3\left[1+\frac{1}{3}\cdot\frac{1}{9}-\frac{1}{9}\left(\frac{1}{9}\right)^2+\frac{5}{81}\left(\frac{1}{9}\right)^3+\ldots\right]=3\left[1+\frac{1}{27}-\left(\frac{1}{27}\right)^2+\ldots\right]$$

$$\approx 3[1+.03704-.001372]=3[1.035668]=3.107004$$

Thus $\sqrt{30}\approx 3.107$

Example 5: Find the coefficient of x^n in the expansion of $\frac{1-x}{(1+x)^2}$

Solution: $\frac{1-x}{(1+x)^2}=(1-x)(1+x)^{-2}$

$$=(-x+1)\left[1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\ldots.+\frac{(-2)(-3)\ldots(-2-r+1)}{r!}x^r+\ldots\right]$$

$$=(-x+1)[1+(-1)2x+(-1)^2 3x^2+\ldots+(-1)^r\times(r+1)x^r+\ldots]$$

$$=(-x+1)[1+(-1)2x+(-1)^2 3x^2+\ldots.+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n+\ldots]$$

coefficient of $x^n=(-1)(-1)^{n-1}n+(-1)^n(n+1)$

$$=(-1)^n n+(-1)^n(n+1)$$

$$=(-1)^n[n+(n+1)]$$

$$=(\quad 1)+.(2\qquad 1)$$

Example 6: If x is so small that its cube and higher power can be neglected, show that

$$\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$$

Solution: $\sqrt{\frac{1-x}{1+x}}=(1\quad x)^{1/2}+(1\quad x)^{-1/2}$

$$=\left[1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\ldots\right]\left[1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2+\ldots\right]$$

$$=\left[1-\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right]\left[1-\frac{1}{2}x+\frac{3}{8}x^2+\ldots\right]$$

$$=\left[\left(1-\frac{1}{2}x+\frac{3}{8}x^2\right)+\left(-\frac{1}{2}x+\frac{1}{4}x^2\right)-\frac{1}{8}x^2+\ldots\right]$$

$$=1-\left(\frac{1}{2}+\frac{1}{2}\right)x+\left(\frac{3}{8}+\frac{1}{4}-\frac{1}{8}\right)x^2+\ldots$$

$$\approx 1-x+\frac{1}{2}x^2$$

Example 7: If *m* and *n* are nearly equal, show that

$$\left(\frac{5m-2n}{3n}\right)^{1/3}\approx\frac{m}{m+2n}+\frac{n+m}{3n}$$

Solution: Put $m = n + h$ (here h is so small that its square and higher powers can be neglected)

$$\text{L.H.S.} = \left(\frac{5m-2n}{3n}\right)^{1/3} \quad \left(\frac{5(n+h)-2n}{3n}\right)^{1/3} \quad \left(\frac{3n+5h}{3n}\right)^{1/3}$$

$$= \left(1+\frac{5h}{3n}\right)^{1/3}$$

$$\approx 1+\frac{5h}{9h} \qquad \text{(neglecting square and higher powers of } h\text{)} \quad \text{(i)}$$

$$\text{R.H.S.} = \frac{m}{m+2n}+\frac{n+m}{3n}$$

$$= \frac{n+h}{3n+h}+\frac{2n+h}{3n}$$

$$= \frac{(n+h)}{3n}\left(\frac{1}{1+\frac{h}{3n}}\right) \quad \left(\frac{2}{3} \quad \frac{h}{3n}\right)$$

$$= (n+h)\frac{1}{3n}\left(1+\frac{h}{3n}\right)^{-1}+\left(\frac{2}{3}+\frac{h}{3n}\right)$$

$$= \left(\frac{1}{3}+\frac{h}{3n}\right)\left(1-\frac{h}{3n}+\ldots\right)+\left(\frac{2}{3}+\frac{h}{3n}\right)$$

$$= \left[\frac{1}{3}+\left(-\frac{h}{9n}+\frac{h}{3n}\right)+\ldots\right]+\frac{2}{3}+\frac{h}{3n}$$

$$\approx 1+\frac{5h}{9n} \quad \text{(neglecting square and higher powers of } h\text{)} \qquad \text{(ii)}$$

From (i) and (ii), we have the result.

Example 8: Identify the series: $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots$ as a binomial expansion and find its sum

Solution: Let the given series be identical with.

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots \qquad \text{(A)}$$

We know that (A) is expansion of $(1+x)^n$ for $|x|<1$ and n is not a positive integer. Now comparing the given series with (A) we get:

$$nx = \frac{1}{3} \qquad \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{3.6} \qquad \text{(ii)}$$

From (i), $x = \frac{1}{3n}$

Now substitution of $x = \frac{1}{3n}$ in (ii) gives

$$\frac{n(n-1)}{2!}.\left(\frac{1}{3n}\right)^2 = \frac{1}{6} \text{ or } \frac{n(n-1)}{2!}.\frac{1}{9n^2} \quad \frac{1}{6}$$

or $\quad n-1 = 3n \Rightarrow n = -\frac{1}{2}$

Putting $n = -\frac{1}{2}$ in (iii), we get

$$x = \frac{1}{3\left(-\frac{1}{2}\right)} = -\frac{2}{3}$$

Thus the given series is the expansion of $\left[1+\left(-\frac{2}{3}\right)\right]^{-1/2}$ or $\left(1-\frac{2}{3}\right)^{-1/2}$

Hence the sum of the given series $= \left(1-\frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-\frac{1}{2}} = (3)^{1/2}$

$= \sqrt{3}$

Example 9: For $y = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{2^2.2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{2^3.3!}\left(\frac{4}{9}\right)^3 + \ldots$

show that $5y^2 + 10y - 4 = 0$

Solution: $y = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3 + \ldots$ (A)

Adding 1 to both sides of (A), we obtain

$$1 + y = 1 + \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3 + \ldots \quad \text{(B)}$$

Let the series on the right side of (B) be identical with

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

which is the expansion of $(1+x)^n$ for $|x| < 1$ and n is not a positive integer. On comparing terms of both the series, we get

$$nx = \frac{1}{2}.\left(\frac{4}{9}\right) \quad \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2 \quad \text{(ii)}$$

From (i), $x = \frac{2}{9n}$ (iii)

Substituting $x = \frac{2}{9n}$ in (ii), we get

$$\frac{n(n-1)}{2}.\left(\frac{2}{9n}\right)^2 = \frac{3}{8}.\frac{16}{81} \quad \text{or} \quad \frac{n(n-1)}{2}.\frac{4}{81n^2} = \frac{3}{8}.\frac{16}{81}$$

or $2(n - 1) = 6n$ or $n - 1 = 3n \Rightarrow n = -\frac{1}{2}$

Putting $n = -\frac{1}{2}$ in (iii), we get

$$x = \frac{2}{9\left(-\frac{1}{2}\right)} = -\frac{4}{9}$$

Thus $1 + y = \left(1 - \frac{4}{9}\right)^{-1/2} = \left(\frac{5}{9}\right)^{-1/2} = \left(\frac{9}{5}\right)^{1/2}$

$= \frac{3}{\sqrt{5}}$

or $\sqrt{5}(1+y) = 3$ (iv)

Squaring both the sides of (iv), we get

$5(1 + 2y + y^2) = 9$

or $5y^2 + 10y - 4 = 0$

Exercise 8.3

1. Expand the following upto 4 terms, taking the values of x such that the expansion in each case is valid.

i) $(1-x)^{1/2}$ ii) $(1+2x)^{-1}$ iii) $(1+x)^{-1/3}$ iv) $(4-3x)^{1/2}$

v) $(8-2x)^{-1}$ vi) $(2-3x)^{-2}$ vii) $\frac{(1-x)^{-1}}{(1+x)^2}$ viii) $\frac{\sqrt{1+2x}}{1-x}$

ix) $\frac{(4+2x)^{1/2}}{2-x}$ x) $(1+x-2x^2)^{\frac{1}{2}}$ xi) $(1-2x+3x^2)^{\frac{1}{2}}$

2. Using Binomial theorem find the value of the following to three places of decimals.

i) $\sqrt{99}$ ii) $(.98)^{\frac{1}{2}}$ iii) $(1.03)^{\frac{1}{3}}$ iv) $\sqrt[3]{65}$

v) $\sqrt[4]{17}$ vi) $\sqrt[5]{31}$ vii) $\dfrac{1}{\sqrt[3]{998}}$ viii) $\dfrac{1}{\sqrt[5]{252}}$

ix) $\dfrac{\sqrt{7}}{\sqrt{8}}$ x) $(.998)^{-\frac{1}{3}}$ xi) $\dfrac{1}{\sqrt[6]{486}}$ xii) $(1280)^{\frac{1}{4}}$

3. Find the coefficient of x^n in the expansion of

i) $\dfrac{1+x^2}{(1+x)^2}$ ii) $\dfrac{(1+x)^2}{(1-x)^2}$ iii) $\dfrac{(1+x)^3}{(1-x)^2}$

iv) $\dfrac{(1+x)^2}{(1-x)^3}$ v) $(1-x+x^2-x^3+\ldots)^2$

4. If x is so small that its square and higher powers can be neglected, then show that

i) $\dfrac{1-x}{\sqrt{1+x}} \approx 1-\dfrac{3}{2}x$ ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\dfrac{3}{2}x$

iii) $\dfrac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x} \simeq \dfrac{1}{4}-\dfrac{17}{384}x$

iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2+\dfrac{25}{4}x$

v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx 4\left(1-\dfrac{5x}{6}\right)$

vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

vii) $\dfrac{\sqrt{4-x}+(8-x)^{1/3}}{(8-x)^{1/3}} \approx 2-\dfrac{1}{12}x$

5. If x is so small that its cube and higher power can be neglected, then show that

i) $\sqrt{1-x-2x^2} \approx 1-\dfrac{1}{2}x-\dfrac{9}{8}x^2$ ii) $\sqrt{\dfrac{1+x}{1-x}} \approx 1+x+\dfrac{1}{2}x^2$

6. If x is very nearly equal 1, then prove that $px^p - qx^q \approx (p-q)x^{p+q}$

7. If $p-q$ is small when compared with p or q, show that

$$\frac{(2n+1)p+(2n-1))q}{(2n-1)p+(2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$

8. Show that $\left[\dfrac{n}{2(n+N)}\right]^{1/2} \approx \dfrac{8n}{9n-N}-\dfrac{n+N}{4n}$ where n and N are nearly equal.

9. Identify the following series as binomial expansion and find the sum in each case.

i) $1-\dfrac{1}{2}\left(\dfrac{1}{4}\right)+\dfrac{1.3}{2!4}\left(\dfrac{1}{4}\right)^2-\dfrac{1.3.5}{3!8}\left(\dfrac{1}{4}\right)^3+\ldots$

ii) $1-\dfrac{1}{2}\left(\dfrac{1}{2}\right)+\dfrac{1.3}{2.4}\left(\dfrac{1}{2}\right)^2-\dfrac{1.3.5}{2.4.6}\left(\dfrac{1}{2}\right)^3+\ldots$

iii) $1+\dfrac{3}{4}+\dfrac{3.5}{4.8}+\dfrac{3.5.7}{4.8.12}+\ldots$

iv) $1-\dfrac{1}{2}.\dfrac{1}{3}+\dfrac{1.3}{2.4}\left(\dfrac{1}{3}\right)^2-\dfrac{1.3.5}{2.4.6}\left(\dfrac{1}{3}\right)^3+\ldots$

10. Use binomial theorem to show that $1+\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.12}+\ldots=\sqrt{2}$

11. If $y=\dfrac{1}{3}+\dfrac{1.3}{2!}\left(\dfrac{1}{3}\right)^2+\dfrac{1.3.5}{3!}\left(\dfrac{1}{3}\right)^3+\ldots$, then prove that $y^2+2y-2-0$

12. If $2y=\dfrac{1}{2^2}+\dfrac{1.3}{2!}.\dfrac{1}{2^4}+\dfrac{1.3.5}{3!}.\dfrac{1}{2^6}+\ldots$ then prove that $4y^2+4y-1=0$

13. If $y=\dfrac{2}{5}+\dfrac{1.3}{2!}.\left(\dfrac{2}{5}\right)^2+\dfrac{1.3.5}{3!}\left(\dfrac{2}{5}\right)^3+\ldots$ then prove that $y^2+2y-4=0$